New Perturbation Bounds Analysis of a Kind of Generalized Saddle Point Systems

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Abstract. In this paper we consider new perturbation bounds analysis of a kind of generalized saddle point systems. We provide perturbation upper bounds for the solutions of generalized saddle point systems, which extend the corresponding results in [W. W. Xu, W. Li, New perturbation analysis for generalized saddle point systems, Calcolo., 46(2009), pp. 25-36] to more general cases.

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1. Introduction

The saddle point system appears in scientific and engineering applications, such as, aeronautics, the mixed finite element solution of the Navier-Stokes, the Maxwell equations, electromagnetics and data fitting et. al. Numerical methods and perturbation bounds analysis for solving the saddle point system studied in some literatures. For details, please see [2-15] and the references therein. Recently, Xu et. al. in [1] considered perturbation bounds of the following generalized saddle point systems:

\[
\begin{pmatrix}
A & B^T \\
B & C
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
f \\
g
\end{pmatrix},
\]

where \(A \in \mathbb{R}^{m \times m}, B \in \mathbb{R}^{n \times m}, \) and \(C \in \mathbb{R}^{n \times n}, \) \(n \leq m \) (possibly \(n \ll m\)). This kind of system arises in many application problems, e.g., see [1]. As we know, a number of literatures deal with the solvers of the saddle point problem (1.1) with \(C \neq 0\). Due to practical applications, perturbation analysis of the saddle point problem (1.1) should be discussed and the perturbation bounds and condition numbers for the system (1.1) are derived.

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In this paper we will extend System (1.1) to the more generalized saddle point system and consider perturbation upper bound for the solutions of this system:

\[
\begin{pmatrix}
A & D \\
B & C \\
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
\end{pmatrix} =
\begin{pmatrix}
f \\
g \\
\end{pmatrix},
\] (1.2)

where \(A \in \mathbb{R}^{m \times m}, B \in \mathbb{R}^{n \times m}, D \in \mathbb{R}^{m \times n} \) and \(C \in \mathbb{R}^{n \times n}, x \in \mathbb{R}^{m}, y \in \mathbb{R}^{n}, n \leq m \) (possibly \(n \ll m\)). Let \(\mathcal{A}\) be the coefficient matrix of (1.2) and assume that \(\mathcal{A}\) is nonsingular. The non-singularity conditions of \(\mathcal{A}\) can be referred in Lemma 2.1 of [15]. Obviously, when \(D = B^T\) in (1.2), System (1.2) reduces to System (1.1). We note that the perturbation bounds analysis for the solutions \(x\) and \(y\) of the system (1.2) have not discussed so far. By this motivation, we will consider this problem in the paper.

Let the perturbed system of (1.2) be as follows:

\[
\begin{pmatrix}
\mathcal{A} + \Delta \mathcal{A} \\
\mathcal{B} + \Delta \mathcal{B} \\
\mathcal{C} + \Delta \mathcal{C} \\
\end{pmatrix}
\begin{pmatrix}
x + \Delta x \\
y + \Delta y \\
\end{pmatrix} =
\begin{pmatrix}
f + \Delta f \\
g + \Delta g \\
\end{pmatrix}.
\]

Throughout the paper, we always assume that

\[
\|\Delta \mathcal{A}\|_F \leq \varepsilon \mathcal{D}_1, \quad \|\Delta \mathcal{B}\|_F \leq \varepsilon \mathcal{D}_2, \quad \|\Delta \mathcal{C}\|_F \leq \varepsilon \mathcal{D}_3,
\]

\[
\|\Delta \mathcal{D}\|_F \leq \varepsilon \sigma_1, \quad \|\Delta f\|_2 \leq \varepsilon \mathcal{D}_4, \quad \|\Delta g\|_2 \leq \varepsilon \mathcal{D}_5,
\]

and let

\[
\delta = (\delta_1, \delta_2, \delta_3)^T, \quad \hat{\delta} = (\hat{\delta}_1, \hat{\delta}_2)^T,
\]

where

\[
\varepsilon > 0, \quad \delta_1 = \sqrt{\mathcal{D}_1^2 + \mathcal{D}_2^2}, \quad \delta_2 = \sqrt{\sigma_1^2 + \mathcal{D}_3^2}, \quad \delta_3 = \sqrt{\mathcal{D}_4^2 + \mathcal{D}_5^2},
\]

\[
\hat{\delta}_1 = \sqrt{\mathcal{D}_1^2 + \mathcal{D}_2^2 + \sigma_1^2 + \mathcal{D}_3^2}, \quad \hat{\delta}_2 = \sqrt{\mathcal{D}_4^2 + \mathcal{D}_5^2}.
\]

Here \(\|\cdot\|_F\) denotes the Frobenius-norm.

The rest of the paper is organized as follows. In Section 2 we give some definitions, notations and useful lemmas to deduce the main results. In Section 3 we give perturbation bounds for the solutions of a kind of generalized saddle point systems. In Section 4 we give numerical examples to illustrate our results.

2. Preliminaries

We briefly give some useful lemmas in order to deduce our main results.

**Lemma 2.1.** If \(\mathcal{A}\) is nonsingular, then

i) \[
\begin{pmatrix}
\Delta x \\
\Delta y \\
\end{pmatrix} = \mathcal{H} \theta + \mathcal{A}^{-1}(P, Q) \begin{pmatrix}
\Delta x \\
\Delta y \\
\end{pmatrix},
\]

ii) \[
\begin{pmatrix}
\Delta x \\
\Delta y \\
\end{pmatrix} = \hat{\mathcal{H}} \hat{\theta} + \mathcal{A}^{-1} \Delta \mathcal{A} \begin{pmatrix}
\Delta x \\
\Delta y \\
\end{pmatrix},
\]