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SOR-like Methods with Optimization Model for Augmented Linear Systems

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Abstract. There has been a lot of study on the SOR-like methods for solving the augmented system of linear equations since the outstanding work of Golub, Wu and Yuan (BIT 41(2001)71-85) was presented fifteen years ago. Based on the SOR-like methods, we establish a class of accelerated SOR-like methods for large sparse augmented linear systems by making use of optimization technique, which will find the optimal relaxation parameter ω by optimization models. We demonstrate the convergence theory of the new methods under suitable restrictions. The numerical examples show these methods are effective.

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1. Introduction

Consider the augmented linear systems of the form

$$\begin{pmatrix} A & B \\ B^T & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix}, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite, $O \in \mathbb{R}^{m\times m}$ is zero, $B \in \mathbb{R}^{n\times m}$ has full column rank, $x, b \in \mathbb{R}^n$, $y, q \in \mathbb{R}^m$, $n \gg m$, and B^T is the transpose of the matrix B. These assumptions guarantee the existence and uniqueness of the solution of the system of linear equations (1.1). For the sake of simplicity, also we can consider the following equivalent form of (1.1)

$$\mathscr{A}u = \begin{pmatrix} A & B \\ -B^T & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ -q \end{pmatrix} = f, \tag{1.2}$$

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A large amount of study has been devoted to the augmented linear systems of the form (1.1) (or (1.2)). The reason for this interest is the fact that such problems appear in many different applications of scientific computing, such as the Karush-Kuhn-Tuker (KKT) conditions for linearly constrained quadratic programming problems, or saddle point problems, or an equilibrium system ([11, 26, 29]), the finite element method for solving the Navier-Stokes equation, or elasticity problems, or second-order elliptic problems ([17, 18]), generalized least squares problems ([20, 30]) and from Lagrange multiplier methods ([19]). See [3] and [10] for a comprehensive summary.

Because of the ubiquitous nature of augmented linear systems, numerical methods and results on the problems have published in a wide variety of books, journals and conference proceedings. Very effective solvers have been developed for the important classes of problems (1.1) (or (1.2)). When the matrix blocks A and B are large and sparse, iterative methods become more attractive than direct methods for solving the augmented linear system (1.1) (or (1.2)). For many years deriving an efficient iterative method based on a splitting of the coefficient matrix $\mathscr{A}$ for solving the system of linear equations (1.1) (or (1.2)) has been an important and active topic. Many iterative methods were proposed for solving the system (1.1) (or (1.2)). There are two sub-approaches to the iterative methods: one is "matrix splitting methods", another is "Krylov subspace methods". See [2, 5, 8–11, 20, 34, 35] for more details. Among these methods, the best known and the oldest iterations are the Uzawa and preconditioned Uzawa methods ([1, 12, 16]), but they are special cases of the SOR-like methods presented in [20]. The SOR-like methods, together with the inexact Uzawa algorithm studied in [16], are usually the methods of choice for solving the augmented linear systems (1.1) (or (1.2)), as they are simple, efficient and require small computer memory. Later, many researchers generalized or modified the SOR-like methods and studied their convergence properties for solving the augmented systems from different view in recent years, we refer to [6, 7, 13–15, 21–23, 25, 31–35] and the references therein.

The idea of minimizing the norm of either the error or the residual so that the numerically optimal value of the iterative parameter is determined, first introduced in [4], used to compute a numerically optimal relaxation parameter for the successive overrelaxation (SOR) iterative methods for solving the system of linear equations. Based on the standard quadratic programming technique, the authors of this paper and their collaborators seem to be the first to introduce the auto-optimal weighting matrices for parallel multisplitting iterative method (see [27]) and be the first to come up with the quasi-Chebyshev accelerated (QCA) method to a convergent splitting iteration (see [28]). The optimal weighting matrices of this multisplitting method and the optimal parameter of the QCA method are generated by optimization models for solving the linear systems. These motivated us to accelerate the SOR-like iterative methods, resulting in a class of new SOR-like methods with optimization model for the augmented systems (1.1) (or (1.2)).

The rest of this paper is organized as follows. In Section 2, we state and briefly summary the existing schemes resulting from the SOR-like methods. In Section 3, we present a class of new SOR-like methods with optimization models and provide its convergence results. Two numerical examples further show the proposed methods are effective than the SOR-like and the SSOR-like methods in Section 4. Finally, we end the paper with a conclusion