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Uniformly Stable Explicitly Solvable Finite Difference Method for Fractional Diffusion Equations

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Abstract. A finite difference scheme for the one-dimensional space fractional diffusion equation is presented and analysed. The scheme is constructed by modifying the shifted Grünwald approximation to the spatial fractional derivative and using an asymmetric discretisation technique. By calculating the unknowns in differential nodal point sequences at the odd and even time levels, the discrete solution of the scheme can be obtained explicitly. We prove that the scheme is uniformly stable. The error between the discrete solution and the analytical solution in the discrete l^2 norm is optimal in some cases. Numerical results for several examples are consistent with the theoretical analysis.

AMS subject classifications: 65M06, 65M12, 65M15

Key words: Finite difference scheme, fractional diffusion equation, uniformly stable, explicitly solvable method, asymmetric technique, error estimate.

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1. Introduction

Fractional differential equations (FDE) have extensive application in areas of physics [2, 4, 5, 16, 35, 41], chemistry [18], hydrology [1, 3, 33, 34] and in finance [29, 30, 32]. In particular, FDE describe anomalous phenomena that cannot be modelled accurately by second-order diffusion equations. Thus in contaminant transport in groundwater flow for example, the solutes moving through aquifers generally do not follow a second-order diffusion equation because of large deviations due to Brownian motion, so a governing equation with fractional-order anomalous diffusion provides a more adequate description [3].

Analytical methods invoking Fourier or Laplace transforms have been developed for FDE in a few cases [28, 39], but numerical methods are usually needed. Numerical solutions have been obtained via finite difference methods [6, 8, 9, 19, 22–25, 36–38], finite element methods [10, 11, 15], the DG method [14] and spectral methods [20, 21, 40]. Discretisation

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procedures and corresponding convergence analysis have been investigated, and in particular a shifted Grünwald discretisation with implicit time-stepping has been shown to be stable, convergent and first-order accurate in the space mesh size [26,27].

Unlike operators in integer-order diffusion equations, fractional diffusion operators are nonlocal and so raise subtle stability issues for corresponding numerical approximations. Numerical methods for FDE tend to yield full coefficient matrices with $\mathscr{O}(K^3)$ computational and $\mathscr{O}(K^2)$ storage costs where K is the number of unknowns, in contrast to numerical methods for second-order diffusion equations that usually generate banded coefficient matrices with $\mathscr{O}(K)$ nonzero entries.

In this article, we present a finite difference scheme to solve the FDE that is constructed by modifying the shifted Grünwald's method [26] with an asymmetric technique [31] and adopting different nodal point stencils at odd and even time levels. We prove that the scheme is uniformly stable. Formally, the scheme is implicit. However, the solution can be obtained explicitly by sequencing the nodal points from one side to the other, and then calculating the unknowns according to the sequences at the odd time levels and calculating the unknowns according to the opposite sequences at even time levels. The error between the numerical and analytical solutions in the discrete l^2 norm is $\mathscr{O}(\Delta t^2 h^{-2(\alpha-1)} + \Delta t + h)$, where $\alpha \in (1,2)$ is the order of the spatial fractional derivative, and h and Δt are the respective space and time mesh sizes. The error estimate is thus optimal, with the same order as the implicit shifted Grünwald finite difference scheme under the condition $\Delta t = \mathscr{O}(h^{\alpha-0.5})$. For $\alpha \leq 1.5$, the condition $\Delta t = \mathscr{O}(h)$ needed to balance the error due to the time and space discretisation is sufficient to verify the optimal error estimate. The asymmetric technique has previously been used to construct parallel algorithms by other researchers — e.g. see [12,13,42,43]. Earlier authors have investigated the stability and shown that the truncation error is $\mathscr{O}(\Delta t h^{-1} + \Delta t + h)$ for parabolic problems [12,13], or exploited the asymmetric technique in real calculations [42,43]. To the best of our knowledge, this article is the first to show that the error between the discrete and the analytical solutions is $\mathscr{O}(\Delta t^2 h^{-2(\alpha-1)})$.

In Section 2, we present our numerical scheme, and show that the discrete solution can be obtained explicitly by sequencing the nodal points apppropriately. In Section 3, we prove that the scheme is uniformly stable, and derive the error estimate in Section 4. In Section 5, numerical experiments are presented to verify the theoretical results. Throughout, C denotes a generic constant that may take different values in different contexts.

2. The Asymmetric Finite Difference Scheme

We consider the following initial-boundary value problem involving a one-dimensional FDE of order α, where $1 < \alpha < 2$ [26,27,33]:

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + f(x,t)\,, \qquad x \in (L,R)\,,\ t \in (0,T]\,, \tag{2.1}$$

$$u(x=L,t) = 0\,,\ \ u(x=R,t) = b_R(t)\,, \qquad t \in (0,T]\,, \tag{2.2}$$

$$u(x,0) = \phi(x)\,, \qquad x \in (L,R)\,. \tag{2.3}$$