Perturbation Bound for the Eigenvalues of a Singular Diagonalizable Matrix

Yimin Wei\(^1,2,*\) and Yifei Qu\(^2\)

\(^1\) School of Mathematical Sciences, Fudan University, Shanghai, 200433, China.
\(^2\) Shanghai Key Laboratory of Contemporary Applied Mathematics, Fudan University, Shanghai, 200433, China.

Received 1 February 2013; Accepted (in revised version) 10 September 2013

Available online 24 February 2014

Abstract. In this short note, we present a sharp upper bound for the perturbation of eigenvalues of a singular diagonalizable matrix given by Stanley C. Eisenstat [3].

AMS subject classifications: 15A09, 65F20

Key words: Bauer-Fike theorem, diagonalizable matrix, group inverse, Jordan canonical form.

1. Introduction

For \(A \in \mathbb{C}^{n \times n}\), the smallest nonnegative integer \(k\) satisfying the rank equation,

\[
\text{rank}(A^k) = \text{rank}(A^{k+1})
\]

is called the index of the matrix \(A\) [1,9]. If \(X \in \mathbb{C}^{n \times n}\) is the unique solution of the three matrix equations

\[
A^{k+1}X = A^k, \quad XAX = X, \quad AX =XA,
\]

we call \(X\) the Drazin inverse \(A^D\). If \(\text{index}(A) = 1\), then the Drazin inverse is reduced to the group inverse denoted by \(A^\dagger\) [1,9].

Let us now recall the classical Bauer-Fike theorem of 1960 and its version from 1999.

Theorem 1.1. (Bauer-Fike Theorem [2, 4]) Let \(A\) be diagonalizable — i.e. \(A = X\Lambda X^{-1}\), where the diagonal matrix \(\Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)\), \(\lambda_i\) is the eigenvalue of \(A\). Let \(E\) be the perturbation of \(A\) and \(\mu\) the eigenvalue of \(A+E\). Then

\[
\min_i |\lambda_i - \mu| \leq \kappa_2(X) \|E\|_2.
\]

*Corresponding author. Email addresses: ymwei@fudan.edu.cn, yimin.wei@gmail.com (Y. Wei), 08302010026@fudan.edu.cn (Y. Qu)
If $A$ is invertible, then

$$\min_{i} \left| \frac{\lambda_i - \mu}{\lambda_i} \right| \leq \kappa_2(X) \| A^{-1} E \|_2,$$

(1.2)

where $\kappa_2(X) = \| X^{-1} \|_2 \| X \|_2$ is the condition number of $X$ with respect to the 2-norm.

Wei et al. [7,8] explored how to extend the classical Bauer-Fike theorem to include the singular case, with the help of the group inverse. Later, Eisenstat [3] gave a different version as follows:

**Theorem 1.2.** Suppose that $A$ is singular diagonalizable — i.e. $A = X \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} X^{-1}$, where $\Lambda_1 = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_r)$, $\lambda_i$ ($i = 1, 2, \cdots, r$) is the nonzero eigenvalue of $A$. Let $E$ be the perturbation of $A$, and $\mu$ the eigenvalue of $A + E$. If $|\mu| > \kappa_2(X) \| E \|_2$, then

$$\min_{i} \left| \frac{\lambda_i - \mu}{\lambda_i} \right| \leq \sqrt{1 + \alpha^2 \kappa_2^2(X)} \| A^T E \|_2,$$

(1.3)

where $\alpha = \kappa_2(X) \| E \|_2 / \sqrt{|\mu|^2 - (\kappa_2(X) \| E \|_2)^2}$.

2. Main Results

In this section, we present our main result that improves the upper bound of Ref. [3].

**Theorem 2.1.** Assume that $A$ is singular diagonalizable and $E$ is the perturbation of $A$, and $\mu$ is the eigenvalue of $A + E$. If $|\mu| > \| X^{-1} (I - AA^T) E X \|_2$. Then

$$\min_{i} \left| \frac{\lambda_i - \mu}{\lambda_i} \right| \leq \sqrt{1 + \beta^2 \| X^{-1} A^T E X \|_2},$$

(2.1)

where $\beta = \| X^{-1} (I - AA^T) E X \|_2 / \sqrt{|\mu|^2 - \| X^{-1} (I - AA^T) E X \|_2^2}$.

**Proof.** Let $A = X \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} X^{-1}$, where $\Lambda_1 = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_r)$ is a nonsingular diagonal matrix. Let $x$ be an eigenvector of $A + E$ associated with $\mu$, and denote

$$X^{-1} E X = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \quad \text{and} \quad X^{-1} x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$ 

Since $\mu x = (A + E) x$,

$$\mu \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mu X^{-1} x = X^{-1} (A + E) X X^{-1} x = \begin{pmatrix} E_{11} + \Lambda_1 & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where $\Lambda_1 = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_r)$. Therefore...