An Artificial Boundary Condition for a Class of Quasi-Newtonian Stokes Flows

Baoqing Liu¹, Qing Chen² and Qikui Du²,*

¹ School of Applied Mathematics, Nanjing University of Finance and Economics, Jiangsu Provincial Key Laboratory for Numerical Simulation of Large Scale Complex Systems, Nanjing 210023, China.
² School of Mathematical Sciences, Nanjing Normal University, Jiangsu Provincial Key Laboratory for Numerical Simulation of Large Scale Complex Systems, Nanjing 210023, China.

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Abstract. An artificial boundary condition method, derived in terms of infinite Fourier series, is applied to solve a class of quasi-Newtonian Stokes flows. Based on the natural boundary reduction involving an artificial condition on the artificial boundary, the coupled variational problem and its numerical solution are obtained. The unique solvability of the continuous and discrete formulations are discussed, and the error analysis for the problem is also considered. Finally, an a posteriori error estimate for the corresponding problem is provided.

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1. Introduction

Interior and exterior nonlinear transmission problems often arise in elasticity [4, 10] and fluid mechanics [11]. The coupled finite element method (FEM) and artificial boundary condition method [8, 9], often called the natural boundary element method [5, 20] or DtN method [6, 13], can be one of the most effective methods to solve exterior nonlinear-linear transmission problems — cf. [2–4, 7, 10, 12, 15, 19] and references therein for more details.

There are several investigations for incompressible materials on bounded domains using finite or mixed finite element methods (e.g. [1,2,14,16–18]), and some on unbounded domains (e.g. [3,10,12]), using coupling methods. The purpose of this work is to investigate a class of quasi-Newtonian Stokes flows where the kinematic viscosity is a nonlinear monotone function of the fluid velocity gradient in the plane.

*Corresponding author. Email addresses: lyberal@163.com (B. Liu), duqikui@nju.edu.cn (Q. Du)
We consider the following configuration. Let $\Omega_0$ be a bounded and simply connected domain in $\mathbb{R}^2$ with a sufficiently smooth boundary $\partial \Omega_0 = \Gamma_0$; and let $\Omega_1$ be the annular region with the boundaries $\Gamma_0$ and $\Gamma_1$, where $\Gamma_1$ is another sufficiently smooth boundary with an interior region that contains $\Omega_0$, and $\Omega^c = \mathbb{R}^2 \setminus (\overline{\Omega_0} \cup \overline{\Omega_1})$. In what follows, $\mathbb{R}^{2 \times 2}$ denotes the space of square matrices of order 2 with real entries, $\mathbb{I}$ denotes the identity matrix of $\mathbb{R}^{2 \times 2}$, and given $\tau \triangleq (\tau_{ij})$, $\sigma \triangleq (\sigma_{ij}) \in \mathbb{R}^{2 \times 2}$ we write

$$\text{tr}(\tau) \triangleq \sum_{i=1}^{2} \tau_{ii}, \quad \sigma : \tau \triangleq \sum_{i,j=1}^{2} \sigma_{ij} \tau_{ij},$$

where $\sigma(u,p) \triangleq (\sigma_{ij}(u,p)) \in \mathbb{R}^{2 \times 2}$ is the Cauchy stress tensor and $\epsilon(u) \triangleq (\epsilon_{ij}(u)) \in \mathbb{R}^{2 \times 2}$ denotes the strain tensor of small deformations with representation $\epsilon_{ij}(u) \triangleq \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. The constitutive equation in $\Omega_1$ is then given by

$$\sigma(u,p) = \psi(\nabla u) \nabla u - pI,$$  \hspace{1cm} (1.1)

where $\psi : \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$ is the nonlinear kinematic viscosity function of the fluid that satisfies the Carreau law for viscoelastic flows $\psi(t) \triangleq \kappa_0 + \kappa_1(1 + t^2)^{(\beta-2)/2}$, $\forall t, \kappa_0 \in \mathbb{R}^{+}$, $\beta \in [1,2]$ — cf. [18]. In passing, we note that Eq. (1.1) reduces to the usual linear model when $\beta = 2$, and that the extension of our approach to kinematic viscosity functions not satisfying Eq. (1.2) or Eq. (1.3) below (which includes the Carreau law with $\kappa_0 = 0$ or $\beta > 2$) will be reported elsewhere.

Let $\psi_{ij} : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ be the mapping defined by $\psi_{ij}(r) \triangleq \psi(|r|) r_{ij}$ for all $r \triangleq (r_{ij}) \in \mathbb{R}^{2 \times 2}$ with $i,j \in \{1,2\}$, and let the mapping $\Phi : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$ be defined by $\Phi(r) \triangleq (\psi_{ij}(r))$ for all $r \in \mathbb{R}^{2 \times 2}$. Then it is easy to check that $\psi$ is of class $C^1$, and there exists $C_1,C_2 > 0$ such that for all $r \triangleq (r_{ij})$, $s \triangleq (s_{ij}) \in \mathbb{R}^{2 \times 2}$ we have

$$|\psi_{ij}(r)| \leq C_1 ||r||_{\mathbb{R}^{2 \times 2}}, \quad \left| \frac{\partial}{\partial r_{kl}} \psi_{ij}(r) \right| \leq C_1, \quad \forall i,j,k,l \in \{1,2\} \hspace{1cm} (1.2)$$

and

$$\sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial r_{kl}} \psi_{ij}(r) s_{ij} s_{kl} \geq C_2 ||s||_{\mathbb{R}^{2 \times 2}}^2. \hspace{1cm} (1.3)$$

Furthermore, Eq. (1.1) can be rewritten as

$$\sigma(u,p) = \Phi(\nabla u) - pI; \hspace{1cm} (1.4)$$

and for a linear elastic material in $\Omega^c$ this reduces to

$$\sigma(u,p) = 2\mu \epsilon(u) - pI, \hspace{1cm} (1.5)$$

where $\mu$ is the familiar Lamé constant.

We now take $[H^1(\Omega_1)]^2 \cap [H^1_{loc}(\Omega^c)]^2$ as the space of functions $v \triangleq \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ defined in $\Omega_1 \cup \Gamma_1 \cup \Omega^c$ such that $v|_{\Omega_i} \in [H^1(\Omega_i)]^2$ and $v|_{\Omega^c} \in [H^1_{loc}(\Omega^c)]^2$. For given $f \in [L^2(\Omega_1)]^2$,
$u_0 \in [H^{1/2}(\Gamma_1)]^2$ and $t_0 \in [H^{-1/2}(\Gamma_1)]^2$ the nonlinear-linear exterior transmission problem can be described as: *Find a vector field $u \in [H^1(\Omega_1)]^2 \cap [H^1_{loc}(\Omega^c)]^2$ and a scalar field $p \in L^2(\mathbb{R}^2 \setminus \Omega_0)$ such that*

$$\begin{align*}
- \nabla \cdot \sigma(u, p) &= f, & \text{in } \Omega_1, \\
- \nabla \cdot \sigma(u, p) &= 0, & \text{in } \Omega^c, \\
u &= 0, & \text{on } \Gamma_0, \\
\nabla \cdot u &= 0, & \text{in } \Omega_1 \cup \Omega^c,
\end{align*}$$

(1.6)

with the transmission conditions

$$u^- = u^+ + u_0, \quad \sigma(u^-, p^-)\nu = \sigma(u^+, p^+)\nu + t_0 \quad \text{on } \Gamma_1,$$

(1.7)

where $u^-, p^-$ and $u^+, p^+$ respectively refer to the approximation to $\Gamma_1$ in $\Omega_1$ and $\Omega^c$, $\nu$ denotes the unit outward normal to $\Gamma_1$.

The rest of our presentation is as follows. In Section 2, we obtain the natural integral equation for the unbounded domain cases, the coupled problem, and its well-posedness. In Section 3, we consider the finite element approximations of the equivalent variational problem and give an error estimate. Finally, in Section 4 we provide our *a posteriori* error estimate for the corresponding problem.

### 2. The Coupled Problem and its Well-Posedness

In this section, we derive the artificial boundary condition for the nonlinear-linear transmission problem in $\mathbb{R}^2$ and $\mathbb{R}^3$, and discuss the coupled problem and its well-posedness.

#### 2.1. Natural integral equation

We introduce an artificial boundary $\Gamma_R$, a circle in $\mathbb{R}^2$ and a sphere in $\mathbb{R}^3$, such that the interior region of $\Gamma_R$ contains $\overline{\Omega_0} \cup \overline{\Omega_1}$; then we let $\Omega_2$ be the annular domain bounded by $\Gamma_1$ and $\Gamma_R$ and $\Omega_e \triangleq \mathbb{R}^2 \setminus (\overline{\Omega_0} \cup \overline{\Omega_1})$, with $\Omega_1 \triangleq \Omega_1 \cup \Gamma_1 \cup \Omega_2$. Thus the original problem (1.6)–(1.8) can be separated into a problem in $\Omega_1$ and a problem in $\Omega_e$, together with continuous conditions on the boundary $\Gamma_R$ — i.e.

$$u^{++} = u^{++}, \quad \sigma(u^{--}, p^{--})\nu = \sigma(u^{++, p^{++}})\nu \quad \text{on } \Gamma_R,$$

(2.1)

where $u^{--}, p^{--}$ and $u^{++}, p^{++}$ respectively refer to the approximation to $\Gamma_R$ in $\Omega_2$ and $\Omega_e$ and $\nu$ denotes the unit outward normal to $\Gamma_R$. 

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The original problem confining in $\Omega_\varepsilon$ can be represented as
\[
\begin{cases}
-\nabla \cdot \sigma(u, p) = 0, & \text{in } \Omega_\varepsilon, \\
\sigma(u, p)v = \mathcal{K}_\infty(u, p), & \text{on } \Gamma_R, \\
\nabla \cdot u = 0, & \text{in } \Omega_\varepsilon, \\
u = \mathcal{O}(1) \text{ and } p = \mathcal{O}\left(\frac{1}{|x|}\right), & \text{as } |x| \to \infty, \ x \in \mathbb{R}^2, \\
u = \mathcal{O}\left(\frac{1}{|x|}\right) \text{ and } p = \mathcal{O}\left(\frac{1}{|x|}\right), & \text{as } |x| \to \infty, \ x \in \mathbb{R}^3.
\end{cases}
\] (2.2)

Then following Refs. [8, 9, 20, 21], using Fourier analysis we can obtain the exact artificial boundary condition in terms of $u|_{\Gamma_R}$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

2.1.1. Natural integral equation in $\mathbb{R}^2$

The exact artificial boundary condition on $\Gamma_R$ in $\mathbb{R}^2$ is
\[
\sigma(u, p)v = \mathcal{K}_\infty(u, p) = T(u) \triangleq \left(T_1(u), T_2(u)\right)^T,
\] (2.3)

where $T$ is defined by
\[
T_i(R, \theta) = \frac{2\mu}{\pi R} \sum_{n=1}^{+\infty} \frac{\partial}{\partial \theta} \int_0^{2\pi} \frac{\cos(\varphi - \theta)}{n} \frac{\partial u_i(R, \varphi)}{\partial \varphi} d\varphi, \ \forall \ i \in \{1, 2\}, \ \theta \in [0, 2\pi].
\]

Combining (2.3) with (1.6), we have
\[
\begin{cases}
-\nabla \cdot \sigma(u, p) = f, & \text{in } \Omega_1 \subseteq \mathbb{R}^2, \\
-\nabla \cdot \sigma(u, p) = 0, & \text{in } \Omega_2 \subseteq \mathbb{R}^2, \\
u = 0, & \text{on } \Gamma_0, \\
\nabla \cdot u = 0, & \text{in } \Omega_1 \setminus \Gamma_1 \subseteq \mathbb{R}^2, \\
\sigma(u, p)v = \mathcal{K}_\infty(u, p), & \text{on } \Gamma_R.
\end{cases}
\] (2.4)

The solution of problem (2.4) and (1.7) is the restriction of the original problem (1.6)--(1.8) on the bounded domain $\Omega_i \subseteq \mathbb{R}^2$.

2.1.2. Natural integral equation in $\mathbb{R}^3$

The exact artificial boundary condition on $\Gamma_R$ in $\mathbb{R}^3$ is
\[
\sigma(u, p)v = -\frac{\mu}{R} \left\{ \sum_{l=0}^{+\infty} \sum_{m=-l}^{l+1} \frac{2l^2 + 4l + 3}{l + 2} A_l^{m+1} + \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} (l + 2) B_l^m \right\} = \mathcal{K}_\infty(u, p),
\] (2.5)
where
\[
A_m^l = \frac{1}{(l+1)(2l+3)} \int_S u(R, \theta, \phi) T_l^m \, ds,
\]
\[
B_m^l = \frac{1}{l(l+1)} \int_S u(R, \theta, \phi) T_l^m \, ds,
\]
\[
C_m^l = \frac{1}{(l+1)(2l-1)} \int_S u(R, \theta, \phi) N_l^m \, ds,
\]
and \(S\) denotes the unit spherical surface, such that the families \(\{I_m^l, T_m^l, N_m^l\}\) defined as in Ref. [21] form an orthogonal basis of \(L^2(S)\).

Combining (2.5) with (1.6), we have
\[
\begin{align*}
-\nabla \cdot \sigma(u, p) &= f, \quad \text{in } \Omega_1 \subseteq \mathbb{R}^3, \\
-\nabla \cdot \sigma(u, p) &= 0, \quad \text{in } \Omega_2 \subseteq \mathbb{R}^3, \\
\nabla \cdot u &= 0, \quad \text{on } \Gamma_0, \\
\nabla \cdot u &= 0, \quad \text{in } \Omega_1 \setminus \Gamma_1 \subseteq \mathbb{R}^3, \\
\sigma(u, p) \nu &= J_{\infty}(u, p), \quad \text{on } \Gamma_R.
\end{align*}
\]

The solution of problem (2.5) and (1.7) is the restriction of the original problem (1.6)–(1.8) on the bounded domain \(\Omega_i \subseteq \mathbb{R}^3\).

### 2.2. The equivalent variational problem and its well-posedness

#### 2.2.1. The equivalent variational problems

Let us now focus on the problem (2.4) and (1.7) in \(\mathbb{R}^2\). We use \(W^{m,p}\) to denote the standard Sobolev spaces, with \(\| \cdot \|\) and \(\cdot \|\) referring to the corresponding norms and semi-norms. In particular, we define \(H^m(\Omega) = W^{m,2}(\Omega), \| \cdot \|_{m,\Omega} = \| \cdot \|_{m,2,\Omega}\) and \(\cdot \|_{m,\Omega} = \cdot \|_{m,2,\Omega}\). Let us also introduce the space
\[
V_\Omega = \{ v \mid v \in H^1(\Omega), v|_{\Gamma_0} = 0 \}, \quad (2.7)
\]
and the corresponding norms
\[
\|v\|^2_{0,\Omega} = \int_\Omega |v|^2 \, dx,
\]
\[
\|v\|_{V_\Omega} = \|v\|^2_{1,\Omega} = \int_\Omega \left( |v|^2 + |\nabla v|^2 \right) \, dx,
\]
\[
|v|^2_{V_\Omega} = |v|^2_{1,\Omega} = \int_\Omega \left( |\nabla v|^2 \right) \, dx.
\]

Furthermore we introduce the spaces \(X_\Omega = V_\Omega \times V_\Omega\) and let
\[
V = \{ (u^-, u^+) \mid (u^-, u^+) \in X_\Omega \times X_\Omega, \ u^- = u^+ + u_0, \text{ on } \Gamma_1 \},
\]
\[
V^* = \{ (u^-, u^+) \mid (u^-, u^+) \in X_\Omega \times X_\Omega, \ u^- = u^+, \text{ on } \Gamma_1 \},
\]
Theorem 2.1. Let \( V, M \) and \( V^* \) be Hilbert spaces and \( A : V \to V' \) and \( B : V \to M' \) be nonlinear and linear operators, respectively; and let \( W \triangleq \ker(B) = \{ v \in V : [B(v), q] = 0, \forall q \in M \} \). Assume that \( A \) is Lipschitz-continuous on \( V \) and that \( A(\bar{\pi} + \cdot) \) is uniformly
The nonlinear operator $D$ we separate the proof of unique solvability into several Lemmas as below.

Since $\psi$ for all $u \in V$ and the following estimates hold:

$$\|A(w) - A(u)\|_{V'} \leq \gamma \|w - u\|_V \quad \forall w, u \in V,$$

$$[A(\overline{w} + w) - A(\overline{u} + u), w - u] \geq \alpha \|w - u\|_V^2 \quad \forall \overline{w} \in V, \ w, u \in W,$$

and we also assume that there exists a positive constant $\beta$ such that

$$\sup_{v \in V', v \neq 0} \frac{|B(v), q|}{\|v\|_{V'}} \geq \beta \|q\|_M \quad \forall q \in M.$$

Then given $(F, G) \in V' \times M'$, there exists a unique $(u, p) \in V \times M$ such that

$$[A(u), v] + [B(v), p] = F(v) \quad \forall v \in V',$$

$$B(u, q) = G(q) \quad \forall q \in M,$$

and the following estimates hold:

$$\|u\|_V \leq \frac{1}{\alpha} \|F\| + \frac{1}{\beta} \left(1 + \frac{\gamma}{\alpha}\right) \|G\|, \quad \|p\|_M \leq \frac{1}{\beta} \left(1 + \frac{\gamma}{\alpha}\right) \left(\|F\| + \frac{\gamma}{\beta} \|G\|\right).$$

We separate the proof of unique solvability into several Lemmas as below.

**Lemma 2.1.** The nonlinear operator $D : V \to V'$ defined by

$$[D(u), v] = \int_{\Omega_1} \Phi(\nabla u_1) : \nabla v_1 \, dx + 2\mu \int_{\Omega_2} \varepsilon(u_2) : \varepsilon(v_2) \, dx$$

for all $u = (u_1, u_2), \ v = (v_1, v_2) \in V$ is Lipschitz-continuous and strongly monotone on $V$.

*Proof.* From the definition of $\Phi$ and $\psi$, we have $\Phi(\nabla u) = \psi(|\nabla u|) \nabla u = (\psi_{ij}(\nabla u))$ for $i, j \in \{1, 2\}$. With $\nabla_{ij} v = \partial v_i / \partial x_j$, for all $u, v \in V$ we can write

$$[D(u), v] = \sum_{i, j=1}^{2} \int_{\Omega_1} \psi_{ij}(\nabla u_1) \nabla_{ij} v_1 \, dx + 2\mu \sum_{i, j=1}^{2} \int_{\Omega_2} \varepsilon_{ij}(u_2) \varepsilon_{ij}(v_2) \, dx.$$

Since $\psi$ is of class $C^1$,

$$\psi_{ij}(\nabla u_1) - \psi_{ij}(\nabla v_1) = \int_{0}^{1} \left\{ \sum_{k, l=1}^{2} \frac{\partial^2 \psi_{ij}}{\partial \delta_{kl}}(\delta(x, t)) \nabla_{kl}(u_1 - v_1) \right\} \, dt$$

with $\delta(x, t) = \nabla v_1 + t \nabla (u_1 - v_1)$. Thus from (1.2) and (2.15) we obtain

$$\|\Phi(\nabla u_1) - \Phi(\nabla v_1)\|_{X_{\Omega_1}'} \leq C_1 \|\nabla (u_1 - v_1)\|_{0, \Omega_1} \leq C_1 \|u_1 - v_1\|_{X_{\Omega_1}},$$

(2.16)
where \( C_1 \) is a positive constant. Then combining (1.3) with (2.15),
\[
\sum_{i,j=1}^{2} \int_{\Omega_1} \left\{ \psi_{ij}(\nabla u_1) - \psi_{ij}(\nabla v_1) \right\} \left\{ \nabla_{ij}(u_1) - \nabla_{ij}(v_1) \right\} \, dx \\
\geq C_2 \| \nabla (u_1 - v_1) \|_{0,\Omega_1}^2 = C_2 |u_1 - v_1|_{X_{\Omega_1}}^2,
\]
where \( C_2 \) is a positive constant.

It is notable that the mapping \( \epsilon : [H^1(\Omega_2)]^2 \to [L^2(\Omega_2)]^4 \) is defined by
\[
\epsilon_{ij}(u_2) = \frac{1}{2} \left( \frac{\partial u_{2i}}{\partial x_j} + \frac{\partial u_{2j}}{\partial x_i} \right)
\]
and satisfies
\[
\sum_{i,j=1}^{2} \left\| \epsilon_{ij}(v_2) \right\|_{0,\Omega_2}^2 \leq \frac{1}{2} \sum_{i,j=1}^{2} \left( \left\| \frac{\partial v_{2i}}{\partial x_j} \right\|_{0,\Omega_2}^2 + \left\| \frac{\partial v_{2j}}{\partial x_i} \right\|_{0,\Omega_2}^2 \right) \\
\leq \| v_2 \|_{X_{\Omega_2}}^2 \quad \forall \ v_2 \in X_{\Omega_2},
\]
by Korn’s inequality, so we also have
\[
\sum_{i,j=1}^{2} \left\| \epsilon_{ij}(v_2) \right\|_{0,\Omega_2}^2 \geq C_3 \| v_2 \|_{X_{\Omega_2}}^2 \quad \forall \ v_2 \in X_{\Omega_2},
\]
where \( C_3 \) is a positive constant.

Now in virtue of the linearity of \( \epsilon \), (2.16), (2.18) and (2.17), (2.19) we can get the desired results. From Refs. [8, 9, 20], for the natural integral equation we have:

**Lemma 2.2.** The linear operator \( \hat{D}, \hat{D}_N : V \to V' \) given by (2.10) and (2.13) satisfies
\[
\left| \langle \hat{D}(u), v \rangle \right| \leq C_4 \| u \|_V \| v \|_V, \quad \langle \hat{D}(u), u \rangle \geq 0;
\]
and for the linear operator \( B \) the inf-sup condition as

**Lemma 2.3.** There exists a positive constant \( \beta \) such that
\[
\sup_{v \in V^*, v \neq 0} \frac{\langle B(v), q \rangle}{\| v \|_{V^*}} \geq \beta \| q \|_M, \quad \forall \ q \in M.
\]

On combining Theorem 2.1 with Lemmas 2.1 to 2.3, we reach the following main theorem on the unique solvability and error estimate of the variational problems (2.8) and (2.14).
Theorem 2.2. There exists a unique solution \((u, p) \in V \times M\) of (2.8) and a unique solution 
\((u^N, p^N) \in V \times M\) of problem (2.14). Furthermore, if \(u|_{Γ'_R} \in [H^{m+\frac{1}{2}}(Γ_R)]^2\) with positive integer \(m\), then the following error estimate holds:

\[
\|u - u^N\|_{V^*} + \|p - p^N\|_{M} \leq \frac{C}{(N + 1)^m} \|u|_{Γ'_R}\|_{[H^{m+\frac{1}{2}}(Γ_R)]^2}.
\] (2.20)

Proof. Although we only show the unique solvability of (2.8), the unique results for the problem (2.14) can be obtained similarly. Let \(A, A_N : V \rightarrow V^*\) be the operator given by

\[ [A(u), v] = [D(u), v] + ⟨\tilde{D}(u), v⟩, \quad [A_N(u), v] = [D(u), v] + ⟨\tilde{D}_N(u), v⟩. \] (2.21)

Then by Lemmas 2.1–2.3, one can verify that the conditions of Theorem 2.1 are satisfied.

It remains to show (2.20). We assume that \((u, p) \in V \times M\) and \((u^N, p^N) \in V \times M\) represent the unique solution of the variational problems (2.8) and (2.14), respectively. For \(u^e = u - u^N\) and \(p^e = p - p^N\), we have \((u^e, p^e) \in V^* \times M\); and from (2.8), (2.14) and (2.21) that

\[
\begin{cases}
[A(u), v] - [A_N(u^N), v] + [B(v), p^e] = 0 & \forall v \in V^*,
[B(u^e), q] = 0 & \forall q \in M.
\end{cases}
\] (2.22)

In particular, we take \(v = u^e\) and \(q = p^e\) in (2.22) such that

\[
[A(u), u^e] - [A_N(u^N), u^e] = 0. \] (2.23)

Then combining Lemmas 2.1–2.2 with (2.8), (2.14) (2.21) and (2.23), we deduce

\[
\alpha \|u^e\|^2_V \leq [A_N(u) - A_N(u^N), u^e] = [A_N(u^N), v] - [A(u), u^e] = ⟨\tilde{D}_N(u), u^e⟩ - ⟨\tilde{D}(u), u^e⟩. \] (2.24)

Following Refs. [8, 9], we have the estimate

\[
[⟨\tilde{D}(u), v⟩ - ⟨\tilde{D}_N(u), v⟩] \leq \frac{C}{(N + 1)^m} \|u|_{Γ'_R}\|_{[H^{m+\frac{1}{2}}(Γ_R)]^2} \|v\|_{V^*},
\] (2.25)

where \(C\) is a positive constant, independent of \(N\) and \(m\). From inequality (2.25), the inequality (2.24) yields

\[
\alpha \|u^e\|^2_V \leq \frac{C}{(N + 1)^m} \|u|_{Γ'_R}\|_{[H^{m+\frac{1}{2}}(Γ_R)]^2} \|u^e\|_{V^*},
\] (2.26)

and hence

\[
\|u^e\|_{V^*} \leq \frac{C}{\alpha(N + 1)^m} \|u|_{Γ'_R}\|_{[H^{m+\frac{1}{2}}(Γ_R)]^2}. \] (2.27)

From (2.8), (2.14) and (2.22), for all \(v \in V^*\) we obtain

\[
[B(v), p^e] = [A_N(u^N), v] - [A(u), v]
= [A_N(u^N), v] - [A_N(u), v] + [A_N(u), v] - [A(u), v]
= [A_N(u^N), v] - [A_N(u), v] + ⟨\tilde{D}_N(u), v⟩ - ⟨\tilde{D}(u), v⟩.
\]
Then from the Lipschitz-continuity of $A_N$ and (2.25), we obtain
\[
\frac{[B(v), p^e]}{\|v\|_{V^*}} \leq \left\{ M \|u^e\|_{V^*} + \frac{C}{(N + 1)^m} \|u^e|_{\Gamma_R}\|_{[H^{m+\frac{1}{2}}(\Gamma_R)]^2} \right\}, \quad \forall v \in V^*,
\]
so combining Lemma 2.3 with (2.28) together with (2.27) we get the desired results. \hfill \Box

3. Finite Element Approximation

In this section, we consider the finite element approximation of (2.14). We first divide $\Omega_1$ and $\Omega_2$ into quasi-uniform triangulation meshes $T_h$ with mesh size $h \in I$, where $I$ is an at most numerable set of indexes, such that the nodes on $\Gamma_1$ are coincident. Let $\mathcal{N}_h$ denote the set of nodes in the domain $\Omega_1 \cup \Omega_2 \cup \Gamma_1 \cup \Gamma_R$; and suppose that $\widetilde{V}_h$, $\widetilde{V}_h^*,$ and $\widetilde{M}_h$ are finite-dimensional subspaces of $V$, $V^*$ and $M$ respectively, where
\[
\widetilde{V}_h = \left\{ (u_h^-, u_h^+) \in X_{\Omega_1,h} \times X_{\Omega_2,h} \mid \forall b \in \mathcal{N}_h \cap \Gamma_1, \ u_h^-(b) = u_h^+(b) + u_0(b) \right\},
\]
\[
\widetilde{V}_h^* = \left\{ (u_h^-, u_h^+) \in X_{\Omega_1,h} \times X_{\Omega_2,h} \mid \forall b \in \mathcal{N}_h \cap \Gamma_1, \ u_h^-(b) = u_h^+(b) \right\},
\]
and a discrete inf-sup condition is satisfied — viz. there exists a positive constant $\beta^*$, independent of $h$, such that
\[
\sup_{v_h \in \widetilde{V}_h^*, v_h \neq 0} \frac{[B(v_h), q_h]}{\|v_h\|_{\widetilde{V}_h^*}} \geq \beta^* \|q_h\|_{\widetilde{M}_h} \quad \forall q_h \in \widetilde{M}_h. \tag{3.1}
\]
The discrete problem corresponding to (2.14) is then: Find $u_h^N = (u_h^{N-}, u_h^{N+}) \in \widetilde{V}_h$ and $p_h^N = (p_h^{N-}, p_h^{N+}) \in \widetilde{M}_h$ such that
\[
\begin{align*}
D_N(u_h^{N-}, v_h) + \widehat{D}_N(u_h^{N+}, v_h) + B(p_h^{N-}, v_h) = L(v_h) & \quad \forall v_h \in \widetilde{V}_h^*, \\
B(q_h, u_h^{N+}) = 0 & \quad \forall q_h \in \widetilde{M}_h.
\end{align*}
\tag{3.2}
\]
For the errors $\|u^N - u_h^N\|_{V^*}$ and $\|p^N - p_h^N\|_M$, by a standard technique for mixed finite element methods [23] we have
\[
\|u^N - u_h^N\|_{V^*} + \|p^N - p_h^N\|_M \leq C_0 \left\{ \inf_{v_h \in \widetilde{V}_h} \|u^N - v_h\|_{V^*} + \inf_{q_h \in \widetilde{M}_h} \|p^N - q_h\|_M \right\}, \tag{3.3}
\]
and for the unique solvability of (3.2) one can refer to Ref. [22] for more details.

We now present the error estimate between the solutions of (2.8) and (3.2).

**Theorem 3.1.** Suppose that $(u, p) \in V \times M$ and $(u_h^N, p_h^N) \in \widetilde{V}_h \times \widetilde{M}_h$ are the solutions of problems (2.8) and (2.14), respectively; and that $u|_{\Gamma_R} \in [H^{m+\frac{1}{2}}(\Gamma_R)]^2$ for some positive integer $m$. Then there exists a positive constant $C$, independent of $h$, $N$ and $m$, such that
\[
\|u - u_h^N\|_{V^*} + \|p - p_h^N\|_M \leq C \left\{ \inf_{v_h \in \widetilde{V}_h} \|u - v_h\|_{V^*} + \inf_{q_h \in \widetilde{M}_h} \|p - q_h\|_M + \frac{1}{(N + 1)^m} \|u|_{\Gamma_R}\|_{[H^{m+\frac{1}{2}}(\Gamma_R)]^2} \right\}. \tag{3.4}
\]
Proof: From the triangle inequality and (3.3),
\[
\|u - u_h^N\|_{V^*} + \|p - p_h^N\|_M \\
\leq \|u - u_h^N\|_{V^*} + \|u_h^N - u_h^N\|_{V^*} + \|p - p_h^N\|_M + \|p_h^N - p_h^N\|_M \\
\leq \|u - u_h^N\|_{V^*} + \|p - p_h^N\|_M + C_0 \left\{ \inf_{v_h \in V_h} \|u_h^N - v_h\|_{V^*} + \inf_{q_h \in M_h} \|p_h^N - q_h\|_M \right\}. \tag{3.5}
\]

Next, we observe that
\[
\inf_{v_h \in V_h} \|u_h^N - v_h\|_{V^*} \leq \|u - u_h^N\|_{V^*} + \inf_{v_h \in V_h} \|u - v_h\|_{V^*}, \tag{3.6}
\]
and
\[
\inf_{q_h \in M_h} \|p_h^N - q_h\|_M \leq \|p - p_h^N\|_M + \inf_{q_h \in M_h} \|p - q_h\|_M. \tag{3.7}
\]

Then combining (3.5)–(3.7) with (2.20), we obtain the desired results. \(\square\)

Furthermore, let us assume the solution of (2.8) is \((u, p)\) with \(u \in X \cap ([H^{m+1}(\Omega_1)]^2 \times [H^{m+1}(\Omega_2)]^2)\) and \(p \in M \cap (H^m(\Omega_1) \times H^m(\Omega_2))\), where \(m\) is a positive integer; and also that the subspaces \(V_h\) and \(M_h\) satisfy the following approximate properties:
\[
\inf_{v_h \in V_h} \|u - v_h\|_{V^*} \leq Ch^m |u|_{k+1, V}, \quad \inf_{q_h \in M_h} \|p - q_h\|_M \leq Ch^m |p|_{k+1, M}. \tag{3.8}
\]

Then from (3.8) the inequality (3.4) can be changed into
\[
\|u - u_h^N\|_{V^*} + \|p - p_h^N\|_M \\
\leq C \left\{ \frac{1}{h^m (|u|_{k+1, V} + |p|_{k, M})} + \frac{1}{(N + 1)^m} \|u|_{r_h} \|_{[H^{m+\frac{1}{2}}(r_h)]^2} \right\}, \tag{3.9}
\]
and if we take \(N = \mathcal{O}(h^{-1})\) then (3.9) can be rewritten as
\[
\|u - u_h^N\|_{V^*} + \|p - p_h^N\|_M \leq Ch^m \left\{ |u|_{k+1, V} + |p|_{k, M} + \|u|_{r_h} \|_{[H^{m+\frac{1}{2}}(r_h)]^2} \right\}. \tag{3.10}
\]
From the error estimate (3.9), it follows that the error can be affected not only by the order of the artificial boundary condition but also by the finite element approximation.

## 4. The a posteriori error estimator

### 4.1. Preliminaries

First, in order to specify \(V_h, V_h^*\) and \(M_h\) we let \(P_1(T)\) and \(P_2(T)\) be defined as in Ref. [12] for each \(T \in T_h\), so we set
\[
\begin{align*}
V_h &= \left\{ (v_h^-, v_h^+) \in \overline{V_h} \mid v_h^+|_T, v_h^-|_T \in [P_2(T)]^2, \forall T \in T_h \right\}, \tag{4.1} \\
V_h^* &= \left\{ (v_h^-, v_h^+) \in \overline{V_h} \mid v_h^+|_T, v_h^-|_T \in [P_2(T)]^2, \forall T \in T_h \right\}, \tag{4.2} \\
M_h &= \left\{ (q_h^-, q_h^+) \in L^2(\Omega_1) \times L^2(\Omega_2) \mid q_h^+|_T, q_h^-|_T \in P_1(T), \forall T \in T_h \right\}. \tag{4.3}
\end{align*}
\]
Then \( V_h \) and \( M_h \) constitute the simplest Hood and Taylor finite element subspaces satisfying the inf-sup condition (3.1), and the approximation properties (3.8) hold.

For simplicity, \( \nu_T \) denotes the unit outward normal, and \( \mathcal{E}(T) \) refers to the set of its edges. With \( \mathcal{E}_h \triangleq \bigcup \{ \mathcal{E}(T) \mid T \in \mathcal{T}_h \} \), this set can be split into the form \( \mathcal{E} \triangleq \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_0) \cup \mathcal{E}_h(\Gamma_R) \), where \( \mathcal{E}_h(\Omega) \triangleq \{ S \mid S \in \mathcal{E}_h, \ S \subseteq \Omega \} \), \( \mathcal{E}_h(\Gamma_0) \triangleq \{ S \mid S \in \mathcal{E}_h, \ S \subseteq \Gamma_0 \} \) and \( \mathcal{E}_h(\Gamma_R) \triangleq \{ S \mid S \in \mathcal{E}_h, \ S \subseteq \Gamma_R \} \). In addition, from Ref. [12] \( X_{\Omega,h} \) possesses the local properties stated below.

**Lemma 4.1.** Let \( \mathcal{I} : V \to V_h \) be the Clément interpolation operator defined as in Ref. [12]. Then there exist two positive constants \( C_1 \) and \( C_2 \), independent of \( h \), such that

\[
\| v - \mathcal{I}(v) \|_{[L^2(T)]^2 \times [L^2(T)]^2} \leq C_1 h_T \| v \|_{[H^1(\Delta(T))]^2 \times [H^1(\Delta(T))]^2}
\]

and

\[
\| v - \mathcal{I}(v) \|_{[L^2(S)]^2} \leq C_2 h_S^{1/2} \| v \|_{[H^1(\Delta(S))]^2}
\]

\( \forall v \in V, T \in \mathcal{T}_h \) and \( S \in \mathcal{E}_h \), where \( h_T(h_S) \) denotes the diameter of \( T \) (\( S \)), \( \Delta(T) \triangleq \bigcup \{ \bar{T} \mid \bar{T} \in \mathcal{T}_h, \ \bar{T} \cap T \neq \emptyset \} \) and \( \Delta(S) \triangleq \bigcup \{ \bar{T} \mid \bar{T} \in \mathcal{T}_h, \ \bar{T} \cap S \neq \emptyset \} \).

It is notable that the family of triangulations \( \{ \mathcal{T}_h \}_{h \in \mathcal{H}} \) is regular, so the numbers of triangles in \( \Delta(T) \) and \( \Delta(S) \) are bounded, independent of \( h \).

**4.2. Main estimates**

We may now provide a reliable a posteriori error estimate. For this purpose, we introduce a functional depending on the mapping \( \Phi \) and artificial boundary condition \( X_{\infty} \).

More precisely, let \( \mathcal{K} : V \to \mathbb{R} \) be defined by

\[
\mathcal{K}(v) \triangleq \int_{\Omega_1} \Phi(\nabla v_1)dx + \mu \int_{\Omega_2} \varepsilon(v_2) : \varepsilon(v_2) + \frac{1}{2}(X_\infty(v_2), v_2)_{\Gamma_h},
\]

where \( v = (v_1, v_2) \in V \). Then we have the following result.

**Lemma 4.2.** The functional \( \mathcal{K} \) has continuous second-order Gâteaux derivatives and there exist two positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 \| v \|_V^2 \leq (\mathcal{D}^2 \mathcal{K})(v) (v, v) \leq C_2 \| v \|_V^2 \quad \forall \ z, v \in V.
\]

**Proof:** From the definition of the Gâteaux derivative, we know the first-order Gâteaux derivative applies \( V \) to its dual and the second-order Gâteaux derivative applies \( V \) to the dual of \( V \times V \) — i.e.

\[
\mathcal{D} \mathcal{K}(z)(v) \triangleq \lim_{t \to 0} \frac{\mathcal{K}(z + tv) - \mathcal{K}(z)}{t} \quad \forall \ z, v \in V,
\]

\[ (4.5) \]
so that
\[
\mathcal{H}(z)(v) = \sum_{i,j=1}^{2} \int_{\Omega_1} \psi_{ij}(\nabla z) \nabla v_i \, dx + 2\mu \sum_{i,j=1}^{2} \int_{\Omega_2} \varepsilon_{ij}(\varepsilon_{ij}(v_2)) \, dx + \left\langle \mathcal{K}_{\infty}(\varepsilon_{ij}(v_2)) \right\rangle,
\]
where \(\psi_{ij}\) and \(\nabla v_i\) are defined in Lemma 2.1, and
\[
\mathcal{H}^2(z)(v, w) = \lim_{t \to 0} \frac{\mathcal{H}(z + tv)(w) - \mathcal{H}(z)(w)}{t} \quad \forall \, z, v, w \in V.
\]
Then it follows from (4.5)–(4.7) that
\[
\mathcal{H}^2(z)(v, w) = \sum_{i,j=1}^{2} \sum_{k,l=1}^{2} \int_{\Omega_1} \frac{\partial \psi_{ij}(\nabla z)}{\partial k} \nabla_{kl} v_i \nabla_{ij} w_j \, dx + 2\mu \sum_{i,j=1}^{2} \int_{\Omega_2} \varepsilon_{ij}(w_2) \, dx + \left\langle \mathcal{K}_{\infty}(w_2), w_2 \right\rangle,
\]
hence from (1.2)–(1.3), Lemmas (2.1) and (2.2) we have (4.4). The continuity of \(\mathcal{H}^2\) follows from (4.8), as the function \(\psi\) is of class \(C^1\).

Now let \((u^N, p^N) \in V \times M\) and \((u^N_h, p^N_h) \in V_h \times M_h\) be the unique solution of the continuous and discrete formulations (2.14) and (3.2), respectively. Then we have the following result.

**Lemma 4.3.** There exists a positive constant \(C_0\), independent of \(h\), such that
\[
\left\| u^N - u^N_h \right\|_V + \left\| p^N - p^N_h \right\|_M \\
\leq C_0 \sup_{(v, q) \in V \times M} \left\{ [D_N(u^N) - D_N(u^N_h), v] + [\tilde{D}_N(u^N - u^N_h), v] \right\} \\
+ \left[ B(v), p^N - p^N_h \right] + \left[ B(u^N - u^N_h), q \right].
\]

**Proof.** From the continuity of \(\mathcal{H}\) there exists \(z^N \in V\), a convex linear combination of \(u^N\) and \(u^N_h\) such that
\[
\mathcal{H}^2(z^N)(u^N - u^N_h, v) = \mathcal{H}(u^N)(v) - \mathcal{H}(u^N_h)(v), \quad \forall \, v \in V^*. \tag{4.10}
\]
From Lemma 2.2 and (4.4),
\[
\mathcal{H}^2(z^N)(v, v) \geq C_1 \left\| v \right\|^2_{V^*}, \quad \forall \, v \in V^*.
\]
Moreover, from Lemma 2.3 the linear operator \(B\) satisfies the continuous inf-sup condition, so Brezzi’s theory implies that there exists a positive constant \(C_0\) such that
\[
\left\| (\tilde{u}, \tilde{p}) \right\|_{V^* \times M} \leq \sup_{(v, q) \in V^* \times M} \left\{ \mathcal{H}^2(z^N)(\tilde{u}, v) + [B(v), \tilde{p}] + [B(\tilde{u}), q] \right\}, \tag{4.11}
\]
for all \((\tilde{u}, \tilde{p}) \in V^* \times M\). In particular, we take \((\tilde{u}, \tilde{p}) = (u^N - u^N_h, p^N - p^N_h)\), when (4.11) becomes

\[
\|u^N - u^N_h\|_{V^*} + \|p^N - p^N_h\|_M 
\leq \sup_{(v, q) \in V^* \times M, \|v, q\| \leq 1} \left\{ 2\mathcal{H}(z^N)(u^N - u^N_h, v) + \left[ B(v), p^N - p^N_h \right] + \left[ B(u^N - u^N_h), q \right] \right\}. \tag{4.12}
\]

According to (2.14), (3.2) and (4.6), we obtain

\[
D_N(u^N, v) + \bar{D}_N(u^N, v) = \mathcal{H}(u^N)(v) \tag{4.13}
\]

and

\[
D_N(u^N_h, v) + \bar{D}_N(u^N_h, v) = \mathcal{H}(u^N_h)(v), \tag{4.14}
\]

hence from (4.10) and (4.12)–(4.14) we get the desired result.

Let us now proceed to estimate the terms on the right-hand side of (4.9). For simplicity, let \(\Theta \equiv [D_N(u^N) - D_N(u^N_h), v] + \langle \bar{D}_N(u^N - u^N_h), v \rangle + [B(v), p^N - p^N_h], \Omega = \Omega_1 \cup \Omega_2, f \equiv \begin{cases} f_1, & \text{in } \Omega_1, \\ 0, & \text{in } \Omega_2, \end{cases}\) and \(\|\sigma(u^N_h, p^N_h)\|_{\nu_T}\) denote the jump of the tractions \(\sigma(u^N_h, p^N_h)\) across the edges on \(T\). From (2.14) and (3.2),

\[
0 = \left[ D_N(u^N_h) - D_N(u^N), v_h \right] + \langle \bar{D}_N(u^N - u^N_h), v_h \rangle + \left[ B(v_h), p^N_h - p^N \right],
\]

so \(\Theta\) can be rewritten as

\[
\Theta = \left[ D_N(u^N) - D_N(u^N_h), v - v_h \right] + \langle \bar{D}_N(u^N - u^N_h), v - v_h \rangle + \left[ B(v - v_h), p^N - p^N_h \right]. \tag{4.15}
\]

Then using \(v = v_h = 0\) on \(\Gamma_0, -\nabla \cdot \sigma(u^N, p^N) = f\) in \(\Omega, \sigma(u^N, p^N)\nu = \mathcal{K}(u^N, p^N)\) and the transmission condition (1.7) on \(\Gamma_1\), similar to the Green’s formula in Ref. [20] one has

\[
\Theta = \sum_{T \in \mathcal{T}_h, \Gamma_T} \int_T \left[ f + \nabla \cdot \sigma(u^N, p^N_h) \right] (v - v_h)dx + \sum_{S \in \mathcal{E}_h(\Gamma_1)} \int_S t_0 (v - v_h)ds
\]

\[
+ \sum_{S \in \mathcal{E}_h(\Gamma_0)} \int_S \left[ \sigma(u^N, p^N)\nu_T - \sigma(u^N_h, p^N_h)\nu_T \right] (v - v_h)ds
\]

\[
+ \sum_{S \in \mathcal{E}_h(\Gamma_1)} \int_S \left[ \sigma(u^N, p^N)\nu - \sigma(u^N_h, p^N_h)\nu \right] (v - v_h)ds. \tag{4.16}
\]

From the continuity of the tractions \(\sigma(u^N, p^N)\nu_T\) through the edges \(S \in \mathcal{E}_h(\Omega),\)

\[
\sum_{S \in \mathcal{E}_h(\Gamma)} \int_S \sigma(u^N, p^N)\nu_T (v - v_h)ds = 0,
\]
such that \((4.16)\) becomes

\[
\Theta = \sum_{T \in \mathcal{T}_h} \int_T \left[ f + \nabla \cdot \sigma(u_h^N, p_h^N) \right] (v - v_h) \, dx + \sum_{S \in \delta(\Gamma)} \int_S t_0(v - v_h) \, ds \\
+ \sum_{S \in \delta(\Omega) \setminus \delta(\Gamma)} \int_S \sigma(u_h^N, p_h^N) \nu_T(v - v_h) \, ds \\
+ \sum_{S \in \delta(\Gamma)} \int_S \left[ \sigma(u_h^N, p_h^N) \nu - \sigma(u_h^N, p_h^N) \nu \right] (v - v_h) \, ds .
\]

Then taking \(v_h = \mathcal{N}_h(v) \in V_h\) in \((4.17)\), from Lemma 4.1 and the Cauchy-Schwarz's inequality we have

\[
\Theta \leq C \sum_{T \in \mathcal{T}_h} \left\{ h_T \left\| f + \nabla \cdot \sigma(u_h^N, p_h^N) \right\|_{[L^2(T)]^2 \times [L^2(T)]^2} \left\| v \right\|_{[H^1(\Delta(T))]^2 \times [H^1(\Delta(T))]^2} \\
+ \sum_{S \in \partial(T) \setminus \delta(\Gamma)} h_S^{1/2} \left\| t_0 \right\|_{[L^2(S)]^2 \times [L^2(S)]^2} \left\| v \right\|_{[H^1(\Delta(S))]^2 \times [H^1(\Delta(S))]^2} \\
+ \sum_{S \in \partial(\Omega) \setminus \delta(\Gamma)} h_S^{1/2} \left\| \sigma(u_h^N, p_h^N) \nu \right\|_{[L^2(S)]^2 \times [L^2(S)]^2} \left\| v \right\|_{[H^1(\Delta(S))]^2 \times [H^1(\Delta(S))]^2} \\
+ \sum_{S \in \partial(\Gamma) \setminus \delta(\Gamma)} h_S^{1/2} \left\| \sigma(u_h^N, p_h^N) \nu - \sigma(u_h^N, p_h^N) \nu \right\|_{[L^2(S)]^2 \times [L^2(S)]^2} \right\} \\
\leq C \eta \sum_{T \in \mathcal{T}_h} \left\{ \left\| v \right\|_{[H^1(\Delta(T))]^2 \times [H^1(\Delta(T))]^2}^2 + \left\| v \right\|_{[H^1(\Delta(S))]^2 \times [H^1(\Delta(S))]^2}^2 \right\}
\]

where \(C\) is a positive constant independent of \(h\), and

\[
\eta^2 = \sum_{T \in \mathcal{T}_h} \left\{ h_T^2 \left\| f + \nabla \cdot \sigma(u_h^N, p_h^N) \right\|_{[L^2(T)]^2 \times [L^2(T)]^2}^2 + \sum_{S \in \partial(T) \setminus \delta(\Gamma)} h_S \left\| t_0 \right\|_{[L^2(S)]^2 \times [L^2(S)]^2}^2 \\
+ \sum_{S \in \partial(\Omega) \setminus \delta(\Gamma)} h_S \left\| \sigma(u_h^N, p_h^N) \nu \right\|_{[L^2(S)]^2 \times [L^2(S)]^2}^2 \\
+ \sum_{S \in \partial(\Gamma) \setminus \delta(\Gamma)} h_S \left\| \sigma(u_h^N, p_h^N) \nu - \sigma(u_h^N, p_h^N) \nu \right\|_{[L^2(S)]^2 \times [L^2(S)]^2}^2 \right\}.
\]

Since the numbers of triangles in \(\Delta(T)\) and \(\Delta(S)\) are bounded independently of \(\mathcal{T}_h\),

\[
\Theta \leq C \eta, \quad \forall \ v \in V, \text{ with } \left\| v \right\| \leq 1 .
\]

On the other hand, from \((2.14)\) one has \([B(u^N), q] = 0\) for all \(q \in M\), so

\[
[B(u^N - u_h^N), q] = - [B(u_h^N), q] = \int_{\Omega} q \nabla \cdot (u_h^N) \, dx,
\]
and hence
\[
\begin{align*}
[B(u^N - u^N_h), q] &\leq \left\| \nabla \cdot (u^N_h) \right\|_{[L^2(\Omega)]^2} \left\| q \right\|_{[L^2(\Omega)]^2} \\
&\leq \left\{ \sum_{T \in \mathcal{T}_h} \left\| \nabla \cdot (u^N_h) \right\|_{[L^2(\Omega)]^2}^2 \right\}^{1/2}
\end{align*}
\] (4.19)
for all \( q \in M \) with \( \left\| q \right\| \leq 1 \). Finally, the following result can be obtained by invoking (4.18) and (4.19) in (4.9).

**Theorem 4.1.** There exists a positive constant \( C \), independent of \( h \), such that

\[
\left\| u^N - u^N_h \right\|_{V^*} + \left\| p^N - p^N_h \right\|_M \leq C \left\{ \sum_{T \in \mathcal{T}_h} \eta_T^2 \right\},
\]

with \( T \in \mathcal{T}_h \) and

\[
\eta_T^2 \equiv h_T^2 \left\| f + \nabla \cdot \sigma(u^N_h, p^N_h) \right\|_{[L^2(T)]^2 \times [L^2(T)]^2}^2 + \sum_{S \in \mathcal{E}(T) \cap \partial_0(T)} h_S \left\| t_0 \right\|_{[L^2(S)]^2 \times [L^2(S)]^2}^2
\]

\[
+ \sum_{S \in \mathcal{E}(T) \cap \partial_0(\Omega)} h_S \left\| \sigma(u^N_h, p^N_h) \nu_T \right\|_{[L^2(S)]^2 \times [L^2(S)]^2}^2 + \left\| \nabla \cdot (u^N_h) \right\|_{[L^2(T)]^2}^2
\]

\[
+ \sum_{S \in \mathcal{E}(T) \cap \partial_0(T_h)} h_S \left\| \sigma(u^N_h, p^N) \nu - \sigma(u^N_h, p^N) \nu \right\|_{[L^2(S)]^2 \times [L^2(S)]^2}^2.
\]

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**References**


An Artificial Boundary Condition for Quasi-Newtonian Stokes Flows


