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On Condition Numbers for the Weighted Moore-Penrose Inverse and the Weighted Least Squares Problem involving Kronecker Products

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Abstract. We establish some explicit expressions for norm-wise, mixed and componentwise condition numbers for the weighted Moore-Penrose inverse of a matrix $A \otimes B$ and more general matrix function compositions involving Kronecker products. The condition number for the weighted least squares problem (WLS) involving a Kronecker product is also discussed.

AMS subject classifications: 65F10

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1. Introduction

Consider the weighted least squares problem (WLS) involving Kronecker products [6, 25]

$$\min_{\mathbf{v}} \|(A \otimes B)\mathbf{v} - \mathbf{c}\|_C \,, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, $A \otimes B \in \mathbb{R}_{nq}^{mp\times nq}$, $\mathbf{c} \in \mathbb{R}^{mp}$, $C = M \otimes P$, $M \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{p\times p}$ are two symmetric positive definite matrices, with $\mathbb{R}^{m\times n}$ and $\mathbb{R}_r^{m\times n}$ respectively denoting the set of all $m \times n$ real matrices and the set of all $m \times n$ real matrices with rank r, and $\mathbb{R}^m = \mathbb{R}^{m\times 1}$. The solution of (1.1) is relevant to the weighted Moore-Penrose inverse involving a Kronecker product. Kronecker products are widely used in system and control theory [7, 8, 26], signal processing [9], image processing [23], computing Markov chains [16], and play an important role in computing the solution of Sylvester matrix equations [14].

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Here we study the condition numbers of the weighted Moore-Penrose inverse and the WLS problem (1.1) involving Kronecker products, important for sensitivity in some computational problems as first discussed by Rice [20]. To take into account the relative scaling of data components or possible sparseness, two kinds of condition numbers have increasingly been considered — viz. mixed condition numbers and component-wise condition numbers [11]. Mixed condition numbers measure errors in the output with norms but the input perturbation component-wise, and component-wise condition numbers measure both the error in the output and the perturbation in the input component-wise.

There are some earlier publications on the condition numbers of the weighted Moore-Penrose inverse involving a Kronecker product and the WLS problem (1.1). Perturbation analysis for the LS problem is discussed in Refs. [2, 3, 5, 21]) for example, and related results on mixed and component-wise condition numbers of the WLS problem in Ref. [17]. Recently, Diao *et al.* [10] presented explicit expression for condition numbers for the linear least squares problem involving Kronecker products.

The rest of this paper is organized as follows. In Section 2, some basic notation and preliminaries are provided. In Section 3, we investigate the norm-wise, mixed, and component-wise condition numbers for the weighted Moore-Penrose inverse involving Kronecker products. In Section 4, we discuss the condition numbers for the associated WLS problem (1.1), and in Section 5 we report some numerical comparisons.

2. Preliminaries

For $A \in \mathbb{R}^{m\times n}$, we denote the transpose of A by A^T, the rank of A by rank(A), and the identity matrix of order n by I_n, respectively. The symbols $\|\cdot\|_F$ and $\|\cdot\|_2$ stand for the Frobenius norm and the spectral norm (or the Euclidean vector norm). For a vector $\mathbf{a} = (a_1, a_2, \cdots, a_n)$, $\|\mathbf{a}\|_\infty$ denotes the infinity norm and $D_{\mathbf{a}} = \mathrm{diag}(a_1, a_2, \cdots, a_n)$. Let $A = [\mathbf{a}_1\ \mathbf{a}_2 \cdots \mathbf{a}_n] \in \mathbb{R}^{m\times n}$, $\mathrm{vec}(A) = [\mathbf{a}_1^T\ \mathbf{a}_2^T \cdots \mathbf{a}_n^T]^T$, and $D_A = D_{\mathrm{vec}(A)}$.

In order to define mixed and component-wise condition numbers, the following form of component-wise distance will be useful — for any $c \in \mathbb{R}$,

$$c^{\ddagger} = \begin{cases} 1/c\,, & \text{if } c \neq 0\,, \\ 1\,, & \text{otherwise.} \end{cases}$$

Furthermore, for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ we define component-wise division by

$$\frac{\mathbf{a}}{\mathbf{b}} = D_{\mathbf{b}}^{\ddagger}\mathbf{a}\,, \tag{2.1}$$

where $D_{\mathbf{b}}^{\ddagger} = \mathrm{diag}(b_1^{\ddagger}, b_2^{\ddagger}, \cdots, b_n^{\ddagger})$. The component-wise distance between $\mathbf{a}$ and $\mathbf{b}$ is then defined by

$$d(\mathbf{a}, \mathbf{b}) = \left\| \frac{\mathbf{a} - \mathbf{b}}{\mathbf{b}} \right\|_\infty = \max_{1\leqslant i\leqslant n} \left\{ |b_i^{\ddagger}||a_i - b_i| \right\}. \tag{2.2}$$

Moreover, we can extend the function d to the matrix case in the following manner. For $A, B \in \mathbb{R}^{m\times n}$, we define A/B by

$$\left(\frac{A}{B}\right)_{ij} = b_{ij}^{\ddagger} a_{ij} ,$$

and the component-wise distance between A and B by

$$d(A,B) = d(\text{vec}(A), \text{vec}(B)) = \left\|\frac{A-B}{B}\right\|_{max} ,$$

where $\|M\|_{max} = \max_{i,j} |m_{ij}|$ for $M = (m_{ij})$. For a vector $\mathbf{a} = [a_1, a_2, \cdots, a_p]^T \in \mathbb{R}^p$, we also define

$$\Omega(\mathbf{a}) = \{k \mid a_k = 0,\ 1 \le k \le p\} .$$

Further, given $\epsilon > 0$ we denote

$$B(\mathbf{a},\epsilon) = \{\mathbf{x} \in \mathbb{R}^p \mid \|\mathbf{x}-\mathbf{a}\|_2 \leqslant \epsilon\|\mathbf{a}\|_2\} ,$$

and

$$B^o(\mathbf{a},\epsilon) = \{\mathbf{x} \mid |x_i - a_i| \leqslant \epsilon|a_i|, i = 1:p\} .$$

It is obvious that if $x \in B^o(\mathbf{a},\varepsilon)$ then $\Omega(\mathbf{a}) \subseteq \Omega(\mathbf{x})$ and $\mathbf{x} = \text{diag}(\mathbf{a})\text{diag}^{\ddagger}(\mathbf{a})\mathbf{x}$.

Definition 2.1. Let X and Y be Banach spaces, and let $S \subset X$ be an open subset of X. A function $F : S \to Y$ is called Fréchet differentiable at $a \in S$ if there exists a bounded linear operator $F_a : S \to Y$ such that

$$\lim_{t\to 0} \frac{\|F(a+t) - F(a) - F_a(t)\|_Y}{\|t\|_X} = 0 ,$$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are norms defined in X and Y, respectively. The linear operator $F_a = F'(a)$ is called the Fréchet derivative of F at a.

Definition 2.2. Let $F : \mathbb{R}^p \to \mathbb{R}^q$ be a continuous mapping defined on an open set $S_F \subset \mathbb{R}^p$ and $\mathbf{a} \in S_F, \mathbf{a} \neq \mathbf{0}$, such that $F(\mathbf{a}) \neq \mathbf{0}$. Then
(i) the norm-wise condition number of F at $\mathbf{a}$ is defined by

$$\kappa(F,\mathbf{a}) = \lim_{\epsilon\to 0} \sup_{\substack{\mathbf{x}\in B(\mathbf{a},\epsilon)\\ \mathbf{x}\neq\mathbf{a}}} \frac{\|F(\mathbf{x}) - F(\mathbf{a})\|_2/\|F(\mathbf{a})\|_2}{\|\mathbf{x}-\mathbf{a}\|_2/\|\mathbf{a}\|_2} ;$$

(ii) the mixed condition number of F at $\mathbf{a}$ is

$$m(F,\mathbf{a}) = \lim_{\varepsilon\to 0} \sup_{\substack{\mathbf{x}\in B^o(\mathbf{a},\varepsilon)\\ \mathbf{x}\neq\mathbf{a}}} \frac{\|F(\mathbf{x}) - F(\mathbf{a})\|_\infty}{\|F(\mathbf{a})\|_\infty} \frac{1}{d(\mathbf{x},\mathbf{a})} ;$$

(iii) the component-wise condition number of F at $\mathbf{a}$ is

$$c(F,\mathbf{a}) = \lim_{\varepsilon\to 0} \sup_{\substack{\mathbf{x}\in B^o(\mathbf{a},\varepsilon)\\ \mathbf{x}\neq\mathbf{a}}} \frac{d(F(\mathbf{x}),F(\mathbf{a}))}{d(\mathbf{x},\mathbf{a})} .$$

Remark 2.1. Definition 2.2 (iii) is the same as that given in Ref. [11] when $F(\mathbf{a})$ has no zero components. Because the proposed distance d is always finite, the hypothesis there that $F(\mathbf{a})$ has no zero components can be removed.

The following lemma provides explicit expressions for the condition numbers given in Definition 2.2.

Lemma 2.1. *Under the same assumptions as in Definition* 2.2*, and supposing that F is Fréchet differentiable at* $\mathbf{a}$*, we have*

$$\kappa(F,\mathbf{a}) = \frac{\|F'(\mathbf{a})\|_2\|\mathbf{a}\|_2}{\|F(\mathbf{a})\|_2}\,,$$
$$m(F,\mathbf{a}) = \frac{\|F'(\mathbf{a})D_{\mathbf{a}}\|_\infty}{\|F(\mathbf{a})\|_\infty} \tag{2.3}$$

and

$$c(F,\mathbf{a}) = \|D^{\ddagger}_{F(\mathbf{a})}F'(\mathbf{a})D_{\mathbf{a}}\|_\infty\,. \tag{2.4}$$

Proof. Here we only prove (2.3), and note that the proof of the other two results are analogous. From Definition 2.2, (2.1) and (2.2) we have

$$\begin{aligned} m(F,\mathbf{a}) &= \lim_{\varepsilon\to 0}\sup_{\substack{\mathbf{x}\in B^o(\mathbf{a},\varepsilon)\\ \mathbf{x}\neq\mathbf{a}}} \frac{\|F(\mathbf{x})-F(\mathbf{a})\|_\infty}{\|F(\mathbf{a})\|_\infty}\frac{1}{d(\mathbf{x},\mathbf{a})} \\ &= \lim_{\varepsilon\to 0}\sup_{\substack{\mathbf{x}\in B^o(\mathbf{a},\varepsilon)\\ \mathbf{x}\neq\mathbf{a}}} \frac{\|F(\mathbf{x})-F(\mathbf{a})\|_\infty}{\|F(\mathbf{a})\|_\infty\|D^{\ddagger}_{\mathbf{a}}(\mathbf{x}-\mathbf{a})\|_\infty}\,. \end{aligned}$$

Since $\mathbf{x}\in B^o(\mathbf{a},\varepsilon)$, we know that $\Omega(\mathbf{a})\subseteq\Omega(\mathbf{x})$ and $\mathbf{x}=D_aD^{\ddagger}_a\mathbf{x}$. Let $\mathbf{y}=D^{\ddagger}_{\mathbf{a}}\mathbf{x}$ and $\mathbf{b}=D^{\ddagger}_{\mathbf{a}}\mathbf{a}$. Then $\mathbf{x}=D_{\mathbf{a}}\mathbf{y}$, $\mathbf{a}=D_{\mathbf{a}}\mathbf{b}$ and $\mathbf{x}\neq\mathbf{a}$ if and only if $\mathbf{y}\neq\mathbf{b}$, so by the Chain Rule of the Fréchet derivative

$$\begin{aligned} m(F,\mathbf{a}) &= \lim_{\varepsilon\to 0}\sup_{\substack{\mathbf{x}\in B^o(\mathbf{a},\varepsilon)\\ \mathbf{x}\neq\mathbf{a}}} \frac{\|F(\mathbf{x})-F(\mathbf{a})\|_\infty}{\|F(\mathbf{a})\|_\infty\|D^{\ddagger}_{\mathbf{a}}\mathbf{x}-D^{\ddagger}_{\mathbf{a}}\mathbf{a}\|_\infty} \\ &= \lim_{\varepsilon\to 0}\sup_{\substack{\mathbf{y}\in B^o(\mathbf{b},\varepsilon)\\ \mathbf{y}\neq\mathbf{b}}} \frac{\|F(D_{\mathbf{a}}\mathbf{y})-F(D_{\mathbf{a}}\mathbf{b})\|_\infty}{\|F(D_{\mathbf{a}}\mathbf{b})\|_\infty\|\mathbf{y}-\mathbf{b}\|_\infty} \\ &= \frac{\|F'(D_{\mathbf{a}}\mathbf{b})D_{\mathbf{a}}\|_\infty}{\|F(D_{\mathbf{a}}\mathbf{b})\|_\infty} \\ &= \frac{\|F'(\mathbf{a})D_{\mathbf{a}}\|_\infty}{\|F(\mathbf{a})\|_\infty}\,. \end{aligned}$$

□

The Kronecker product $A\otimes B$ of $A\in\mathbb{R}^{m\times n}$ and $B\in\mathbb{R}^{p\times q}$ is defined by (e.g. see Ref. [12])

$$A\otimes B=\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B\\ a_{21}B & a_{22}B & \cdots & a_{2n}B\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B\end{bmatrix}\in\mathbb{R}^{mp\times nq}.$$

The following well known properties of the Kronecker product will be used below.

Lemma 2.2. (cf. Refs [13,22]) *Let $A\in\mathbb{R}^{m\times n}$, $B\in\mathbb{R}^{p\times q}$ and $\mathbf{y}\in\mathbb{R}^n$. Then*

$$\begin{aligned} &|A\otimes B|=|A|\otimes|B|,\\ &\|A\otimes B\|_2=\|A\|_2\|B\|_2,\\ &\mathrm{vec}(AXB)=(B^T\otimes A)\mathrm{vec}(X),\\ &\mathrm{vec}(A\otimes B)=(I_n\otimes\Pi_{qm}\otimes I_p)(\mathrm{vec}(A)\otimes\mathrm{vec}(B)),\\ &\mathbf{a}\otimes\mathbf{b}=\mathrm{vec}(\mathbf{b}\mathbf{a}^{\mathbf{T}}),\\ &\Pi_{mn}\mathrm{vec}(A)=\mathrm{vec}(A^T),\\ &\Pi_{mn}(\mathbf{y}\otimes\mathbf{A})=\mathbf{A}\otimes\mathbf{y}, \end{aligned} \tag{2.5}$$

where Π_{mn} is the vec-perturbation matrix

$$\Pi_{mn}=\sum_{i=1}^{m}\sum_{j=1}^{n}E_{ij}(m\times n)\otimes E_{ji}(n\times m). \tag{2.6}$$

Here $E_{ij}(m\times n)=\mathbf{e}_i^{(m)}(\mathbf{e}_j^{(n)})^T\in\mathbb{R}^{m\times n}$, and $\mathbf{e}_i^{(k)}\in\mathbb{R}^k$ is the ith column of the identity matrix of order k.

Next we consider the condition numbers for the weighted Moore-Penrose inverse involving Kronecker products. We recall that there exists a unique matrix $X\in\mathbb{R}^{n\times m}$ such that the following equations hold (e.g. see Ref. [12]):

$$AXA=A,\qquad XAX=X,\qquad (MAX)^T=MAX,\qquad (NXA)^T=NXA.$$

The matrix $X=A_{MN}^{\dagger}$ is said to be the weighted Moore-Penrose inverse of A with respect to the symmetric positive definite matrices M and N, respectively.

3. Weighted Moore-Penrose Inverses involving Kronecker Products

In this section, we present the norm-wise, mixed and component-wise condition numbers in the context of the weighted Moore-Penrose inverse of $A\otimes B$. Consider a matrix function of a matrix X as follows [12]:

$$\mathbb{f}(X)=\begin{bmatrix} f_{11}(X) & f_{12}(X) & \cdots & f_{1t}(X)\\ f_{21}(X) & f_{22}(X) & \cdots & f_{2t}(X)\\ \vdots & \vdots & \ddots & \vdots\\ f_{s1}(X) & f_{s2}(X) & \cdots & f_{st}(X)\end{bmatrix}.$$

Let $X = A \otimes B$, where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$. Then we obtain

$$\mathbb{f}(A\otimes B) = \begin{bmatrix} f_{11}(A\otimes B) & f_{12}(A\otimes B) & \cdots & f_{1t}(A\otimes B) \\ f_{21}(A\otimes B) & f_{22}(A\otimes B) & \cdots & f_{2t}(A\otimes B) \\ \vdots & \vdots & \ddots & \vdots \\ f_{s1}(A\otimes B) & f_{s2}(A\otimes B) & \cdots & f_{st}(A\otimes B) \end{bmatrix}.$$

Henceforth we always assume that the matrices $\tilde{M}$, $\tilde{N}$, $\tilde{P}$ and $\tilde{Q}$ are all positive definite, and $\tilde{C} = \tilde{M} \otimes \tilde{P}$ and $\tilde{D} = \tilde{N} \otimes \tilde{Q}$. According to Definition 2.2, the norm-wise, mixed and component-wise condition numbers for the weighted Moore-Penrose inverse of $\mathbb{f}(A\otimes B)$ with respect to $\tilde{C}$ and $\tilde{D}$ can be defined as

$$\kappa\left(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)\right) = \lim_{\epsilon\to 0} \sup_{\substack{\sqrt{\|\Delta A\|_F^2+\|\Delta B\|_F^2} \\ \leqslant \epsilon\sqrt{\|A\|_F^2+\|B\|_F^2}}} \frac{\left\|\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}((A+\Delta A)\otimes(B+\Delta B)) - \mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)\right\|_F}{\epsilon\left\|\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)\right\|_F},$$

$$m\left(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)\right) = \lim_{\epsilon\to 0} \sup_{\substack{\|\frac{\Delta A}{A}\|_\infty\leqslant\epsilon \\ \|\frac{\Delta B}{B}\|_\infty\leqslant\epsilon}} \frac{\left\|vec(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}((A+\Delta A)\otimes(B+\Delta B)) - \mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B))\right\|_\infty}{\epsilon\left\|vec(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B))\right\|_\infty},$$

$$c\left(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)\right) = \lim_{\epsilon\to 0} \sup_{\substack{\|\frac{\Delta A}{A}\|_\infty\leqslant\epsilon \\ \|\frac{\Delta B}{B}\|_\infty\leqslant\epsilon}} \frac{1}{\epsilon}\left\|\frac{\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}((A+\Delta A)\otimes(B+\Delta B)) - \mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)}{\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)}\right\|_\infty,$$

respectively. The following lemma is useful in the sequel.

Lemma 3.1. (cf. Refs [19,22]) *Let* $\mathbf{x} = [x_1,\ x_2,\cdots,x_n]^T \in \mathbb{R}^n$ *and* $F(\mathbf{x}) = [f_1, f_2,\cdots,f_m]^T$, *where each* f_i *is a real-valued differentiable function of* $\mathbf{x}$. *Then the matrix representation of* $F'(\mathbf{x})$ *is given by the Jacobian matrix*

$$F'(\mathbf{x}) = \frac{\partial F(\mathbf{x})}{\partial \mathbf{x}^T} = \begin{bmatrix} \frac{\partial}{\partial x_1}f_1(\mathbf{x}) & \frac{\partial}{\partial x_2}f_1(\mathbf{x}) & \cdots & \frac{\partial}{\partial x_n}f_1(\mathbf{x}) \\ \frac{\partial}{\partial x_1}f_2(\mathbf{x}) & \frac{\partial}{\partial x_2}f_2(\mathbf{x}) & \cdots & \frac{\partial}{\partial x_n}f_2(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1}f_m(\mathbf{x}) & \frac{\partial}{\partial x_2}f_m(\mathbf{x}) & \cdots & \frac{\partial}{\partial x_n}f_m(\mathbf{x}) \end{bmatrix}.$$

Lemma 3.2. *Let* X *be an* $m \times n$ *matrix of full rank. Then the differential* $dX^{\dagger}_{MN}$ *is*

$$\begin{aligned} dX^{\dagger}_{MN} =& (I_n - X^{\dagger}_{MN}X)N^{\dagger}(dX^T){X^{\dagger}_{MN}}^T NX^{\dagger}_{MN} \\ &+ X^{\dagger}_{MN}M^{\dagger}{X^{\dagger}_{MN}}^T(dX^T)M(I_m - XX^{\dagger}_{MN}) - X^{\dagger}_{MN}(dX)X^{\dagger}_{MN}, \end{aligned}$$

so the Jacobian matrix is

$$\begin{aligned} \frac{\partial \text{vec}(X^{\dagger}_{MN})}{\partial \text{vec}(X)^T} =& \Big\{({X^{\dagger}_{MN}}^T NX^{\dagger}_{MN}) \otimes (I_n - X^{\dagger}_{MN}X)N^{\dagger} + M(I_m - XX^{\dagger}_{MN}) \otimes \\ & X^{\dagger}_{MN}M^{\dagger}{X^{\dagger}_{MN}}^T\Big\}\Pi_{mn} - {X^{\dagger}_{MN}}^T \otimes X^{\dagger}_{MN}, \end{aligned}$$

where Π_{mn} *is given by* (2.6).

Proof. From Theorem 8.3 of Ref. [22],

$$dX^\dagger = (I_n - X^\dagger X)(dX^T){X^\dagger}^T X^\dagger + X^\dagger {X^\dagger}^T (dX^T)(I_m - XX^\dagger) - X^\dagger (dX) X^\dagger.$$

Since $X_{MN}^\dagger = N^{-\frac{1}{2}}(M^{\frac{1}{2}}XN^{-\frac{1}{2}})^\dagger M^{\frac{1}{2}}$,

$$\begin{aligned}
&dX_{MN}^\dagger = d\left(N^{-\frac{1}{2}}(M^{\frac{1}{2}}XN^{-\frac{1}{2}})^\dagger M^{\frac{1}{2}}\right) = N^{-\frac{1}{2}} d\left((M^{\frac{1}{2}}XN^{-\frac{1}{2}})^\dagger\right) M^{\frac{1}{2}}\\
=&N^{-\frac{1}{2}}\left(I_n - (M^{\frac{1}{2}}XN^{-\frac{1}{2}})^\dagger M^{\frac{1}{2}}XN^{-\frac{1}{2}}\right)\left(d(M^{\frac{1}{2}}XN^{-\frac{1}{2}})^T\right){(M^{\frac{1}{2}}XN^{-\frac{1}{2}})^\dagger}^T (M^{\frac{1}{2}}XN^{-\frac{1}{2}})^\dagger M^{\frac{1}{2}}\\
&+N^{-\frac{1}{2}}(M^{\frac{1}{2}}XN^{-\frac{1}{2}})^\dagger {(M^{\frac{1}{2}}XN^{-\frac{1}{2}})^\dagger}^T\left(d(M^{\frac{1}{2}}XN^{-\frac{1}{2}})^T\right)\left(I_m - M^{\frac{1}{2}}XN^{-\frac{1}{2}}(M^{\frac{1}{2}}XN^{-\frac{1}{2}})^\dagger\right)M^{\frac{1}{2}}\\
&-N^{-\frac{1}{2}}(M^{\frac{1}{2}}XN^{-\frac{1}{2}})^\dagger\left(d(M^{\frac{1}{2}}XN^{-\frac{1}{2}})\right)(M^{\frac{1}{2}}XN^{-\frac{1}{2}})^\dagger M^{\frac{1}{2}}\\
=&(I_n - X_{MN}^\dagger X)N^\dagger(dX^T){X_{MN}^\dagger}^T N X_{MN}^\dagger + X_{MN}^\dagger M^\dagger {X_{MN}^\dagger}^T (dX^T) M (I_m - XX_{MN}^\dagger)\\
&-X_{MN}^\dagger (dX) X_{MN}^\dagger
\end{aligned}$$

and

$$\begin{aligned}
&d(\mathrm{vec}(X_{MN}^\dagger)) = \mathrm{vec}\left(d(X_{MN}^\dagger)\right)\\
=&\Big\{\left[({X_{MN}^\dagger}^T N X_{MN}^\dagger)\otimes(I_n - X_{MN}^\dagger X)N^\dagger + M(I_m - XX_{MN}^\dagger)\otimes X_{MN}^\dagger M^\dagger {X_{MN}^\dagger}^T\right]\Pi_{mn}\\
&-{X_{MN}^\dagger}^T\otimes X_{MN}^\dagger\Big\} d(\mathrm{vec(X)})\,.
\end{aligned}$$

This completes the proof of the lemma. □

According to Lemma 3.2, we consider the Fréchet derivative of the mapping $(\mathbf{a},\mathbf{b}) \longmapsto \mathrm{vec}(\mathbb{f}_{\tilde{C}\tilde{D}}^\dagger)$, where $\mathbf{a} = \mathrm{vec}(A), \mathbf{b} = \mathrm{vec}(B), \mathbb{f} = \mathbb{f}(A\otimes B)$ and $\mathbb{f}_{\tilde{C}\tilde{D}}^\dagger = \mathbb{f}_{\tilde{C}\tilde{D}}^\dagger(A\otimes B)$.

Lemma 3.3. *Let $A\in\mathbb{R}^{m\times n}, B\in\mathbb{R}^{p\times q}$, and the mapping $\psi:\mathbb{R}^{mn}\times\mathbb{R}^{pq}\longmapsto\mathbb{R}^{st}$ be $\psi(\mathbf{a},\mathbf{b}) =$* $\mathrm{vec}(\mathbb{f}_{\tilde{C}\tilde{D}}^\dagger(A\otimes B))$, *where* $\mathbf{a} = \mathrm{vec}(A), \mathbf{b} = \mathrm{vec}(B)$. *If $\mathbb{f}$ is continuously differentiable at $A\otimes B$ and $\mathbb{f}(A\otimes B)$ has full rank, then ψ is continuous and Fréchet differentiable. Furthermore,*

$$\psi'(\mathbf{a},\mathbf{b}) = \begin{bmatrix}\phi_{(A,B)}L_B & \phi_{(A,B)}L_A\end{bmatrix}\,,$$

where $\phi_{(A,B)}$, L_A, L_B, $\widetilde{\Pi}$ are defined by (3.1) *and* (3.2) *in the proof below.*

Proof. From Lemma 3.2 and that $\mathbb{f}(A\otimes B)$ has full rank, $\mathbb{f}_{\tilde{C}\tilde{D}}^\dagger(A\otimes B)$ is differentiable at $\mathbb{f}(A\otimes B)$. Since $\mathbb{f}$ is continuously differentiable at $A\otimes B$, $\psi = \mathrm{vec}\circ\mathbb{f}_{\tilde{C}\tilde{D}}^\dagger$ is differentiable with respect to $[(\mathrm{vec}(A))^T\ (\mathrm{vec}(B))^T]^T$. From Lemma 3.2,

$$\begin{aligned}
d(\mathbb{f}_{\tilde{C}\tilde{D}}^\dagger(A\otimes B)) =&(I_t - \mathbb{f}_{\tilde{C}\tilde{D}}^\dagger\mathbb{f})\tilde{D}^\dagger\left(d(\mathbb{f}(A\otimes B))^T\right){\mathbb{f}_{\tilde{C}\tilde{D}}^\dagger}^T\tilde{D}\mathbb{f}_{\tilde{C}\tilde{D}}^\dagger\\
&+\mathbb{f}_{\tilde{C}\tilde{D}}^\dagger\tilde{C}^\dagger{\mathbb{f}_{\tilde{C}\tilde{D}}^\dagger}^T\left(d(\mathbb{f}(A\otimes B))^T\right)\tilde{C}(I_s - \mathbb{f}\mathbb{f}_{\tilde{C}\tilde{D}}^\dagger) - \mathbb{f}_{\tilde{C}\tilde{D}}^\dagger d\left(\mathbb{f}(A\otimes B)\right)\mathbb{f}_{\tilde{C}\tilde{D}}^\dagger
\end{aligned}$$

and

$$d\left(\mathrm{vec}(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B))\right) = \mathrm{vec}(d(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)))$$
$$= \Big\{\Big[{\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}}^{T}\tilde{D}\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}\otimes(I_t-\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}\mathbb{f})\tilde{D}^{\dagger}+\tilde{C}(I_s-\mathbb{f}\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}})\otimes\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}\tilde{C}^{\dagger}{\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}}^{T}\Big]\Pi_{st}$$
$$-({\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}}^{T}\otimes\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}})\Big\}\mathrm{vec}\left(d(\mathbb{f}(A\otimes B))\right).$$

From Lemma 3.2 of Ref. [10],

$$\mathrm{vec}(d(\mathbb{f}(A\otimes B))) = \left(\frac{\partial\,\mathbb{f}(A\otimes B)}{\partial\,\mathrm{vec}(A\otimes B)^T}\right)(I_n\otimes\Pi_{qm}\otimes I_p)$$
$$\times\left\{\left(I_{mn}\otimes\mathrm{vec}(B)\right)\Pi_{mn}\mathrm{vec}(dA)+\left(\mathrm{vec}(A)\otimes I_{pq}\right)\mathrm{vec}(dB)\right\}.$$

Let

$$L_A=\mathrm{vec}(A)\otimes I_{pq},\qquad L_B=\left(I_{mn}\otimes\mathrm{vec}(B)\right)\Pi_{mn},\qquad \widetilde{\Pi}=I_n\otimes\Pi_{qm}\otimes I_p\,. \tag{3.1}$$

and

$$\phi_{(A,B)}=\Big\{\Big[{\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}}^{T}\tilde{D}\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}\otimes(I_t-\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}\mathbb{f})\tilde{D}^{\dagger}+\tilde{C}(I_s-\mathbb{f}\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}})\otimes\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}\tilde{C}^{\dagger}{\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}}^{T}\Big]\Pi_{st}-({\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}}^{T}\otimes\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}})\Big\}$$
$$\times\left(\frac{\partial\,\mathbb{f}(A\otimes B)}{\partial\,\mathrm{vec}(\mathrm{A}\otimes\mathrm{B})^{\mathrm{T}}}\right)\widetilde{\Pi}\,. \tag{3.2}$$

Then we can derive

$$d(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B))=\begin{bmatrix}\phi_{(A,B)}L_B & \phi_{(A,B)}L_A\end{bmatrix}\begin{bmatrix}d(\mathrm{vec}(A))\\ d(\mathrm{vec}(B))\end{bmatrix}.$$

□

From the Fréchet derivative of $\psi(\mathbf{a},\mathbf{b})$, we obtain the following explicit expressions of the norm-wise, mixed, and component-wise condition numbers for the weighted Moore-Penrose inverse of $\mathbb{f}(A\otimes B)$ with respect matrices $\tilde{C}$ and $\tilde{D}$.

Theorem 3.1. *Under the same assumptions as in Lemma* 3.3*, we have*

$$\kappa\left(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)\right)=\frac{\left\|[\phi_{(A,B)}L_B\quad \phi_{(A,B)}L_A]\right\|_2\sqrt{\|A\|_F^2+\|B\|_F^2}}{\left\|\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)\right\|_F},$$
$$m\left(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)\right)=\frac{\left\||\phi_{(A,B)}L_B|\mathrm{vec}(|A|)+|\phi_{(A,B)}L_A|\mathrm{vec}(|B|)\right\|_\infty}{\left\|\mathrm{vec}(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B))\right\|_\infty}, \tag{3.3}$$

and

$$c\left(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)\right)=\left\|\frac{|\phi_{(A,B)}L_B|\mathrm{vec}(|A|)+|\phi_{(A,B)}L_A|\mathrm{vec}(|B|)}{\mathrm{vec}(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B))}\right\|_\infty. \tag{3.4}$$

Proof. Here we only prove the first equation and (3.4). The proof of (3.3) is analogous. According to Lemmas 2.1 and 3.3, we have

$$\kappa\left(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)\right)=\frac{\|\psi'(\mathbf{a},\mathbf{b})\|_2\|(\mathbf{a}^T,\mathbf{b}^T)\|_2}{\|\psi(\mathbf{a},\mathbf{b})\|_2}=\frac{\left\|\left[\phi_{(A,B)}L_B \quad \phi_{(A,B)}L_A\right]\right\|_2\sqrt{\|\mathbf{a}\|_2^2+\|\mathbf{b}\|_2^2}}{\left\|\mathrm{vec}(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B))\right\|_2}$$
$$=\frac{\left\|\left[\phi_{(A,B)}L_B \quad \phi_{(A,B)}L_A\right]\right\|_2\sqrt{\|A\|_F^2+\|B\|_F^2}}{\left\|\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)\right\|_F}.$$

From (2.4), we have

$$\begin{aligned}
c\left(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)\right)&=\left\|D^{\ddagger}_{\psi(\mathbf{a},\mathbf{b})}\psi'(\mathbf{a},\mathbf{b})D_{A,B}\right\|_\infty\\
&=\left\|D^{\ddagger}_{\mathrm{vec}(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B))}\left[\phi_{(A,B)}L_B \quad \phi_{(A,B)}L_A\right]\begin{bmatrix}D_A & \\ & D_B\end{bmatrix}\right\|_\infty\\
&=\left\|\left|D^{\ddagger}_{\mathrm{vec}(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B))}\left[\phi_{(A,B)}L_B \quad \phi_{(A,B)}L_A\right]\begin{bmatrix}D_A & \\ & D_B\end{bmatrix}\right|\mathbf{e}\right\|_\infty\\
&=\left\|\frac{\left|\left[\phi_{(A,B)}L_B \quad \phi_{(A,B)}L_A\right]\right|\begin{bmatrix}\mathrm{vec}(|A|)\\ \mathrm{vec}(|B|)\end{bmatrix}}{\mathrm{vec}\left(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)\right)}\right\|_\infty\\
&=\left\|\frac{\left|\phi_{(A,B)}L_B\right|\mathrm{vec}(|A|)+\left|\phi_{(A,B)}L_A\right|\mathrm{vec}(|B|)}{\mathrm{vec}\left(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)\right)}\right\|_\infty,
\end{aligned}$$

where $D_{A,B}=\begin{bmatrix}D_A & \\ & D_B\end{bmatrix}$ with D_A (and similarly D_B) as in the definition in the first paragraph of Section 2, and $\mathbf{e}=[1,1,\cdots,1]^T$. □

Let $\mathbb{g}(A):\mathbb{R}^{m\times n}\longmapsto\mathbb{R}^{e\times f}$ and $\mathbb{h}(B):\mathbb{R}^{p\times q}\longmapsto\mathbb{R}^{g\times h}$ be two matrix functions and $\mathbb{f}(A\otimes B)=\mathbb{g}(A)\otimes\mathbb{h}(B)$. Henceforth we always assume that $C=M\otimes P$, $D=N\otimes Q$, where M, N, P and Q be positive definite matrices of orders m, n, p and q, respectively. From Theorem 2.1 of Ref. [24],

$$(A\otimes B)^{\dagger}_{CD}=A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ}.$$

It is then easy to obtain $\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)=\mathbb{g}^{\dagger}_{\tilde{M}\tilde{N}}(A)\otimes\mathbb{h}^{\dagger}_{\tilde{P}\tilde{Q}}(B)$, where $\tilde{C}=\tilde{M}\otimes\tilde{P}$, $\tilde{D}=\tilde{N}\otimes\tilde{Q}$, $\tilde{M}$, $\tilde{N}$, $\tilde{P}$ and $\tilde{Q}$ are positive definite matrices of orders e, f, g and h, respectively.

Using Lemmas 2.2 and 3.2, we obtain

$$
\begin{aligned}
& d\left(\mathrm{vec}\big(\mathbb{f}^{\dagger}_{\tilde{C}\tilde{D}}(A\otimes B)\big)\right) \\
=& d\left(\mathrm{vec}\big(\mathrm{g}^{\dagger}_{\tilde{M}\tilde{N}}(A)\otimes \mathbb{h}^{\dagger}_{\tilde{P}\tilde{Q}}(B)\big)\right) \\
=& \mathrm{vec}\left(d\left(\mathrm{g}^{\dagger}_{\tilde{M}\tilde{N}}(A)\right)\otimes \mathbb{h}^{\dagger}_{\tilde{P}\tilde{Q}}(B)+\mathrm{g}^{\dagger}_{\tilde{M}\tilde{N}}(A)\otimes d\left(\mathbb{h}^{\dagger}_{\tilde{P}\tilde{Q}}(B)\right)\right) \\
=& \widehat{\Pi}\left\{\mathrm{vec}\big(d\left(\mathrm{g}^{\dagger}_{\tilde{M}\tilde{N}}(A)\right)\big)\otimes \mathrm{vec}\big(\mathbb{h}^{\dagger}_{\tilde{P}\tilde{Q}}(B)\big)+\mathrm{vec}\big(\mathrm{g}^{\dagger}_{\tilde{M}\tilde{N}}(A)\big)\otimes \mathrm{vec}\big(d\left(\mathbb{h}^{\dagger}_{\tilde{P}\tilde{Q}}(B)\right)\big)\right\} \\
=& \widehat{\Pi}\left\{\left(I_{fe}\otimes \mathrm{vec}\big(\mathbb{h}^{\dagger}_{\tilde{P}\tilde{Q}}(B)\big)\right)\mathrm{vec}\big(d\left(\mathrm{g}^{\dagger}_{\tilde{M}\tilde{N}}(A)\right)\big)+\left(\mathrm{vec}\big(\mathrm{g}^{\dagger}_{\tilde{M}\tilde{N}}(A)\big)\otimes I_{hg}\right)\mathrm{vec}\big(d\left(\mathbb{h}^{\dagger}_{\tilde{P}\tilde{Q}}(B)\right)\big)\right\} \\
=& \widehat{\Pi}\Big\{\left(I_{fe}\otimes \mathrm{vec}\big(\mathbb{h}^{\dagger}_{\tilde{P}\tilde{Q}}(B)\big)\right)\Big\{\Big[\big({\mathrm{g}^{\dagger}_{\tilde{M}\tilde{N}}}^{T}(A)N\mathrm{g}^{\dagger}_{\tilde{M}\tilde{N}}(A)\big)\otimes \big(I_f-\mathrm{g}^{\dagger}_{\tilde{M}\tilde{N}}(A)\mathrm{g}(A)\big)\tilde{N}^{\dagger} \\
& +\tilde{M}\left(I_e-\mathrm{g}(A)\mathrm{g}^{\dagger}_{\tilde{M}\tilde{N}}(A)\right)\otimes \mathrm{g}^{\dagger}_{\tilde{M}\tilde{N}}(A)\tilde{M}^{\dagger}{\mathrm{g}^{\dagger}_{\tilde{M}\tilde{N}}}^{T}(A)\Big]\Pi_{ef}-{\mathrm{g}^{\dagger}_{\tilde{M}\tilde{N}}}^{T}(A)\otimes \mathrm{g}^{\dagger}_{\tilde{M}\tilde{N}}(A)\Big\}\mathrm{vec}\big(d(\mathrm{g}(A))\big) \\
& +\left(\mathrm{vec}(\mathrm{g}^{\dagger}_{\tilde{M}\tilde{N}}(A))\otimes I_{hg}\right)\Big\{\Big[\big({\mathbb{h}^{\dagger}_{\tilde{P}\tilde{Q}}}^{T}(B)\tilde{Q}\mathbb{h}^{\dagger}_{\tilde{P}\tilde{Q}}(B)\big)\otimes \big(I_h-\mathbb{h}^{\dagger}_{\tilde{P}\tilde{Q}}(B)\mathbb{h}(B)\big)\tilde{Q}^{\dagger} \\
& +\tilde{P}\left(I_g-\mathbb{h}(B)\mathbb{h}^{\dagger}_{\tilde{P}\tilde{Q}}(B)\right)\otimes \mathbb{h}^{\dagger}_{\tilde{P}\tilde{Q}}(B)\tilde{P}^{\dagger}{\mathbb{h}^{\dagger}_{\tilde{P}\tilde{Q}}}^{T}(B)\Big]\Pi_{gh}-{\mathbb{h}^{\dagger}_{\tilde{P}\tilde{Q}}}^{T}(B)\otimes \mathbb{h}^{\dagger}_{\tilde{P}\tilde{Q}}(B)\Big\}\mathrm{vec}\big(d(\mathbb{h}(B))\big)\Big\},
\end{aligned}
\tag{3.5}
$$

where $\widehat{\Pi}=(I_e\otimes \Pi_{gf}\otimes I_h)$.From (3.5), we have the following corollary.

Corollary 3.1. *Let* $A\in\mathbb{R}^{m\times n}_{n}$, $B\in\mathbb{R}^{p\times q}_{q}$ *and* $\psi(\mathbf{a},\mathbf{b})=\mathrm{vec}((A\otimes B)^{\dagger}_{CD})$, *where* $\mathbf{a}=\mathrm{vec}(A)$ *and* $\mathbf{b}=\mathrm{vec}(B)$. *Then* ψ *is continuous and Fréchet differentiable at all* $(\mathbf{a},\mathbf{b})$. *Furthermore,*

$$
\psi'(\mathbf{a},\mathbf{b})=\begin{bmatrix}Q_BM_A & P_AN_B\end{bmatrix},
$$

where

$$
\begin{aligned}
M_A &= -({A^{\dagger}_{MN}}^{T}\otimes A^{\dagger}_{MN})+\left[M(I_m-AA^{\dagger}_{MN})\otimes (A^TMA)^{-1}\right]\Pi_{mn}, \\
P_A &= (I_m\otimes \Pi_{pn}\otimes I_q)\left(\mathrm{vec}(A^{\dagger}_{MN})\otimes I_{qp}\right), \\
N_B &= -({B^{\dagger}_{PQ}}^{T}\otimes B^{\dagger}_{PQ})+\left[P(I_p-BB^{\dagger}_{PQ})\otimes (B^TPB)^{-1}\right]\Pi_{pq}, \\
Q_B &= (I_m\otimes \Pi_{pn}\otimes I_q)\left(I_{nm}\otimes \mathrm{vec}(B^{\dagger}_{PQ})\right).
\end{aligned}
$$

Proof. Since A and B have full column rank,

$$
A^{\dagger}_{MN}A=I_n,\qquad B^{\dagger}_{PQ}B=I_q. \tag{3.6}
$$

Let $\mathbb{f}(A\otimes B)=A\otimes B$, $\mathrm{g}(A)=A$ and $\mathbb{h}(B)=B$. Then the result follows from (3.6).

From the Fréchet derivative of $\psi(\mathbf{a},\mathbf{b})$ and Theorem 3.1, we can obtain the explicit expressions of the condition numbers for the weighted Moore-Penrose inverse of $A\otimes B$.

Theorem 3.2. *Under the same assumptions as in Corollary* 3.1*, we have*

$$\kappa\left((A\otimes B)^{\dagger}_{CD}\right)=\frac{\left\|[Q_BM_A \quad P_AN_B]\right\|_2\sqrt{\|A\|_F^2+\|B\|_F^2}}{\left\|A^{\dagger}_{MN}\right\|_F\left\|B^{\dagger}_{PQ}\right\|_F},$$

$$m\left((A\otimes B)^{\dagger}_{CD}\right)=\frac{\left\||Q_BM_A|\mathrm{vec}(|A|)+|P_AN_B|\mathrm{vec}(|B|)\right\|_\infty}{\left\|\mathrm{vec}\left(A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ}\right)\right\|_\infty} \tag{3.7}$$

and

$$c\left((A\otimes B)^{\dagger}_{CD}\right)=\left\|\frac{|Q_BM_A|\mathrm{vec}(|A|)+|P_AN_B|\mathrm{vec}(|B|)}{\mathrm{vec}\left(A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ}\right)}\right\|_\infty. \tag{3.8}$$

Proof. Here we only prove (3.8). The proofs of the other two equations of this theorem are analogous. Let $\mathbb{f}(A\otimes B)=A\otimes B$. According to Theorem 3.1 and (3.5), we have

$$c((A\otimes B)^{\dagger}_{CD})=\left\|\frac{|\phi_{(A,B)}L_B|\mathrm{vec}(|A|)+|\phi_{(A,B)}L_A|\mathrm{vec}(|B|)}{\mathrm{vec}\left(A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ}\right)}\right\|_\infty. \tag{3.9}$$

From Lemma 3.3 and Corollary 3.1,

$$\phi_{(A,B)}L_B=Q_BM_A, \qquad \phi_{(A,B)}L_A=P_AN_B\,, \tag{3.10}$$

hence the result follows from (3.9) and (3.10) . □

Remark 3.1. When $C=I$ and $D=I$,

$$M_A=-(A^{\dagger^T}\otimes A^{\dagger})+\left[(I_m-AA^{\dagger})\otimes(A^TA)^{-1}\right]\Pi_{mn}\,,$$
$$N_B=-(B^{\dagger^T}\otimes B^{\dagger})+\left[(I_p-BB^{\dagger})\otimes(B^TB)^{-1}\right]\Pi_{pq}\,,$$
$$P_A=(I_m\otimes\Pi_{pn}\otimes I_q)\left(\mathrm{vec}(A^{\dagger})\otimes I_{qp}\right)\,,$$

and

$$Q_B=(I_m\otimes\Pi_{pn}\otimes I_q)\left(I_{nm}\otimes\mathrm{vec}(B^{\dagger})\right)\,.$$

Thus the equations in Theorem 3.2 are the same as $\kappa((A\otimes B)^{\dagger}), m((A\otimes B)^{\dagger})$ and $c((A\otimes B)^{\dagger})$ in Ref. [10].

In Theorem 3.2, we present explicit expressions for the condition numbers $\kappa((A\otimes B)^{\dagger}_{CD})$, $m((A\otimes B)^{\dagger}_{CD})$ and $c((A\otimes B)^{\dagger}_{CD})$. However, the vec-perturbation matrix Π is involved in the explicit expression of these condition numbers and is very difficult to compute, and we present some computable upper bounds as follows.

Corollary 3.2. *Under the same assumptions as in Corollary* 3.1*, we have*

$$\kappa\left((A\otimes B)^{\dagger}_{CD}\right) \leq \frac{\|B^{\dagger}_{PQ}\|_F\left(\|A^{\dagger}_{MN}\|_2^2+\|M(I_m-AA^{\dagger}_{MN})\|_2\|(A^TMA)^{-1}\|_2\right)\sqrt{\|A\|_F^2+\|B\|_F^2}}{\|A^{\dagger}_{MN}\|_F\|B^{\dagger}_{PQ}\|_F}$$
$$+\frac{\|A^{\dagger}_{MN}\|_F\left(\|B^{\dagger}_{PQ}\|_2^2+\|P(I_p-BB^{\dagger}_{PQ})\|_2\|(B^TPB)^{-1}\|_2\right)\sqrt{\|A\|_F^2+\|B\|_F^2}}{\|A^{\dagger}_{MN}\|_F\|B^{\dagger}_{PQ}\|_F}$$
$$:=\kappa\left((A\otimes B)^{\dagger}_{CD}\right)^{upper},$$

$$m\left((A\otimes B)^{\dagger}_{CD}\right) \leq \frac{\left\||A^{\dagger}_{MN}|\otimes\left(|B^{\dagger}_{PQ}||B||B^{\dagger}_{PQ}|+|(B^TPB)^{-1}||B|^T|P(I_p-BB^{\dagger}_{PQ})|\right)\right\|_{max}}{\|A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ}\|_{max}}$$
$$+\frac{\left\|\left(|A^{\dagger}_{MN}||A||A^{\dagger}_{MN}|+|(A^TMA)^{-1}||A|^T|M(I_m-AA^{\dagger}_{MN})|\right)\otimes|B^{\dagger}_{PQ}|\right\|_{max}}{\|A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ}\|_{max}}$$
$$:=m\left((A\otimes B)^{\dagger}_{CD}\right)^{upper},$$

$$c\left((A\otimes B)^{\dagger}_{CD}\right) \leq \left\|\frac{|A^{\dagger}_{MN}|\otimes\left(|B^{\dagger}_{PQ}||B||B^{\dagger}_{PQ}|+|(B^TPB)^{-1}||B|^T|P(I_p-BB^{\dagger}_{PQ})|\right)}{A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ}}\right\|_{max}$$
$$+\left\|\frac{\left(|A^{\dagger}_{MN}||A||A^{\dagger}_{MN}|+|(A^TMA)^{-1}||A|^T|M(I_m-AA^{\dagger}_{MN})|\right)\otimes|B^{\dagger}_{PQ}|}{A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ}}\right\|_{max}$$
$$:=c\left((A\otimes B)^{\dagger}_{CD}\right)^{upper}.$$

Proof. From Lemma 2.2, we have

$$\begin{aligned}\|Q_BM_A\|_2 &\leqslant \|Q_B\|_2\left\|-({A^{\dagger}_{MN}}^T\otimes A^{\dagger}_{MN})+\left[M(I_m-AA^{\dagger}_{MN})\otimes(A^TMA)^{-1}\right]\Pi_{mn}\right\|_2\\ &\leqslant \|\mathrm{vec}(B^{\dagger}_{PQ})\|_2\left(\|{A^{\dagger}_{MN}}^T\otimes A^{\dagger}_{MN}\|_2+\|M(I_m-AA^{\dagger}_{MN})\otimes(A^TMA)^{-1}\|_2\right)\\ &= \|B^{\dagger}_{PQ}\|_F\left(\|A^{\dagger}_{MN}\|_2^2+\|M(I_m-AA^{\dagger}_{MN})\|_2\|(A^TMA)^{-1}\|_2\right),\end{aligned}$$

and

$$\|P_AN_B\|_2 \leqslant \|A^{\dagger}_{MN}\|_F\left(\|B^{\dagger}_{PQ}\|_2^2+\|P(I_p-BB^{\dagger}_{PQ})\|_2\|(B^TPB)^{-1}\|_2\right).$$

Furthermore, we have

$$\begin{aligned}\|[Q_BM_A\quad P_AN_B]\|_2 \leqslant&\|Q_BM_A\|_2+\|P_AN_B\|_2\\ \leqslant&\|B^{\dagger}_{PQ}\|_F\left(\|A^{\dagger}_{MN}\|_2^2+\|M(I_m-AA^{\dagger}_{MN})\|_2\|(A^TMA)^{-1}\|_2\right)\\ &+\|A^{\dagger}_{MN}\|_F\left(\|B^{\dagger}_{PQ}\|_2^2+\|P(I_p-BB^{\dagger}_{PQ})\|_2\|(B^TPB)^{-1}\|_2\right).\end{aligned}$$

According to Theorem 3.2, we can obtain the upper bound $\kappa((A\otimes B)^{\dagger}_{CD})^{upper}$. Then from Lemma 2.2 we obtain

$$
\begin{aligned}
&\left|Q_B M_A\right|\left|\mathrm{vec}(A)\right| \\
\leqslant & \left|\left(I_m\otimes \Pi_{pn}\otimes I_q\right)\left(I_{nm}\otimes \mathrm{vec}\left(B^{\dagger}_{PQ}\right)\right)\right| \\
&\times\left|-({A^{\dagger}_{MN}}^T\otimes A^{\dagger}_{MN})+\left[M\left(I_m-AA^{\dagger}_{MN}\right)\otimes\left(A^TMA\right)^{-1}\right]\Pi_{mn}\right|\left|\mathrm{vec}(A)\right| \\
\leqslant & \left(I_m\otimes \Pi_{pn}\otimes I_q\right)\left(I_{nm}\otimes \mathrm{vec}(|B^{\dagger}_{PQ}|)\right) \\
&\times\left(\left(|{A^{\dagger}_{MN}}^T|\otimes|A^{\dagger}_{MN}|\right)+\left(|M(I_m-AA^{\dagger}_{MN})|\otimes|(A^TMA)^{-1}|\right)\Pi_{mn}\right)\left|\mathrm{vec}(A)\right| \\
\leqslant & \left(I_m\otimes \Pi_{pn}\otimes I_q\right)\left(I_{nm}\otimes \mathrm{vec}(|B^{\dagger}_{PQ}|)\right) \\
&\times\mathrm{vec}\left(|A^{\dagger}_{MN}||A||A^{\dagger}_{MN}|+|(A^TMA)^{-1}||A|^T|M(I_m-AA^{\dagger}_{MN})|\right) \\
\leqslant & \left(I_m\otimes \Pi_{pn}\otimes I_q\right)\mathrm{vec}\Big(\mathrm{vec}(|B^{\dagger}_{PQ}|)\mathrm{vec}\left(|A^{\dagger}_{MN}||A||A^{\dagger}_{MN}|+|(A^TMA)^{-1}||A|^T|M(I_m-AA^{\dagger}_{MN})|\right)^T\Big) \\
= & \left(I_m\otimes \Pi_{pn}\otimes I_q\right)\left[\mathrm{vec}\left(|A^{\dagger}_{MN}||A||A^{\dagger}_{MN}|+|(A^TMA)^{-1}||A|^T|M(I_m-AA^{\dagger}_{MN})|\right)\otimes\mathrm{vec}(|B^{\dagger}_{PQ}|)\right] \\
= & \mathrm{vec}\left(\left(|A^{\dagger}_{MN}||A||A^{\dagger}_{MN}|+|(A^TMA)^{-1}||A|^T|M(I_m-AA^{\dagger}_{MN})|\right)\otimes|B^{\dagger}_{PQ}|\right).
\end{aligned}
$$

We also have

$$
|P_AN_B||\mathrm{vec}(B)|\leqslant \mathrm{vec}\left(|A^{\dagger}_{MN}|\otimes\left(|B^{\dagger}_{PQ}||B||B^{\dagger}_{PQ}|+|(B^TPB)^{-1}||B|^T|P(I_p-BB^{\dagger}_{PQ})|\right)\right).
$$

From Theorem 3.2, $\|\mathrm{vec}(A)\|_\infty=\|A\|_{max}$ and the matrix norm triangular inequality, we can obtain the upper bounds $m((A\otimes B)^{\dagger}_{CD})^{upper}$ and $c((A\otimes B)^{\dagger}_{CD})^{upper}$. □

Remark 3.2. When $C=I$ and $D=I$, we have

$$
\begin{aligned}
\kappa\left((A\otimes B)^{\dagger}\right)^{upper}=&\frac{\|B^{\dagger}\|_F\left(\|A^{\dagger}\|_2^2+\|I_m-AA^{\dagger}\|_2\|(A^TA)^{-1}\|_2\right)\sqrt{\|A\|_F^2+\|B\|_F^2}}{\|A^{\dagger}\|_F\|B^{\dagger}\|_F} \\
&+\frac{\|A^{\dagger}\|_F\left(\|B^{\dagger}\|_2^2+\|I_p-BB^{\dagger}\|_2\|(B^TB)^{-1}\|_2\right)\sqrt{\|A\|_F^2+\|B\|_F^2}}{\|A^{\dagger}\|_F\|B^{\dagger}\|_F}, \\
m\left((A\otimes B)^{\dagger}\right)^{upper}=&\frac{\left\||A^{\dagger}|\otimes\left(|B^{\dagger}||B||B^{\dagger}|+|(B^TB)^{-1}||B|^T|I_p-BB^{\dagger}|\right)\right\|_{max}}{\|A^{\dagger}\otimes B^{\dagger}\|_{max}} \\
&+\frac{\left\|\left(|A^{\dagger}||A||A^{\dagger}|+|(A^TA)^{-1}||A|^T|I_m-AA^{\dagger}|\right)\otimes|B^{\dagger}|\right\|_{max}}{\|A^{\dagger}\otimes B^{\dagger}\|_{max}}, \\
c\left((A\otimes B)^{\dagger}\right)^{upper}=&\left\|\frac{|A^{\dagger}|\otimes\left(|B^{\dagger}||B||B^{\dagger}|+|(B^TB)^{-1}||B|^T|I_p-BB^{\dagger}|\right)}{A^{\dagger}\otimes B^{\dagger}}\right\|_{max} \\
&+\left\|\frac{\left(|A^{\dagger}||A||A^{\dagger}|+|(A^TA)^{-1}||A|^T|I_m-AA^{\dagger}|\right)\otimes|B^{\dagger}|}{A^{\dagger}\otimes B^{\dagger}}\right\|_{max}.
\end{aligned}
$$

It is seen that in this case the results of Corollary 3.2 are the same as $m((A\otimes B)^{\dagger})^{upper}$ and $c((A\otimes B)^{\dagger})^{upper}$ in Ref. [10].

4. Weighted Least Squares Problems (WLS) involving Kronecker Products

Let $A\in\mathbb{R}_n^{m\times n}$, $\mathbf{b}\in\mathbb{R}^m$, $M\in\mathbb{R}^{m\times m}$ and $N\in\mathbb{R}^{n\times n}$ be positive definite matrices. It is well known that the WLS problem

$$\min_{\mathbf{u}\in\mathbb{R}^n}\|A\mathbf{u}-\mathbf{b}\|_M \tag{4.1}$$

has a unique minimum-norm (N) and least-squares (M) solution [18]

$$\mathbf{x}=A_{MN}^{\dagger}\mathbf{b}\,. \tag{4.2}$$

Let us now discuss the condition numbers for the following KPWLS problem:

$$\min_{\mathbf{v}\in\mathbb{R}^{nq}}\left\|(A\otimes B)\mathbf{v}-\mathbf{c}\right\|_C\,, \tag{4.3}$$

where $A\in\mathbb{R}_n^{m\times n}$, $B\in\mathbb{R}_q^{p\times q}$, $A\otimes B\in\mathbb{R}_{nq}^{mp\times nq}$, $\mathbf{c}\in\mathbb{R}^{mp}$, $C=M\otimes P$, $D=N\otimes Q$, and M, N, P and Q are positive definite matrices of orders m, n, p and q, respectively. From (4.2), we know that (4.3) has a unique minimum-norm (D) and least-squares (C) solution

$$\mathbf{x}=(A\otimes B)_{CD}^{\dagger}\mathbf{c}=(A_{MN}^{\dagger}\otimes B_{PQ}^{\dagger})\mathbf{c}=\left((A^TMA)^{-1}\otimes(B^TPB)^{-1}\right)(A^T\otimes B^T)(M\otimes P)\mathbf{c}\,. \tag{4.4}$$

We proceed to generalise some results for the nonsingular linear equations $(A\otimes B)\mathbf{x}=\mathbf{d}$ to the WLS problem involving Kronecker products [15,27].

The perturbed system of (4.3) is

$$\min_{\mathbf{v}\in\mathbb{R}^{nq}}\left\|\left[(A+\Delta A)\otimes(B+\Delta B)\right]\mathbf{v}-(\mathbf{c}+\Delta\mathbf{c})\right\|_C\,, \tag{4.5}$$

where ΔA, ΔB and $\Delta\mathbf{c}$ have the same dimensions as A, B and $\mathbf{c}$, respectively. Let the mapping $\varphi:\mathbb{R}^{mn}\times\mathbb{R}^{pq}\times\mathbb{R}^{mp}\longmapsto\mathbb{R}^{nq}$ be given by $\varphi(\mathbf{a},\mathbf{b},\mathbf{c})=\mathrm{vec}((A\otimes B)_{CD}^{\dagger}\mathbf{c})$, where $\mathbf{a}=\mathrm{vec}(A)$ and $\mathbf{b}=\mathrm{vec}(B)$. Similar to Corollary 3.1, we first consider the Fréchet derivative of φ.

Lemma 4.1. *Let $A\in\mathbb{R}_n^{m\times n}$, $B\in\mathbb{R}_q^{p\times q}$, $\mathbf{c}\in\mathbb{R}^{mp}$, $C=M\otimes P$ and $D=N\otimes Q$, where M, N, P and Q be positive definite matrices of orders m, n, p and q, respectively. Consider $\varphi(\mathbf{a},\mathbf{b},\mathbf{c})=\mathrm{vec}((A\otimes B)_{CD}^{\dagger}\mathbf{c})$, where $\mathbf{a}=\mathrm{vec}(A)$ and $\mathbf{b}=\mathrm{vec}(B)$. Then φ is continuous and Fréchet differentiable at all $(\mathbf{a},\mathbf{b},\mathbf{c})$. Furthermore,*

$$\varphi'(\mathbf{a},\mathbf{b},\mathbf{c})=\left[\mathscr{Q}\;\mathscr{P}\;A_{MN}^{\dagger}\otimes B_{PQ}^{\dagger}\right]\,,$$

where

$$\begin{aligned}
\mathscr{Q}=&\left(\mathbf{r}^T\otimes\left((A^TMA)^{-1}\otimes(B^TPB)^{-1}\right)\right)\left(I_m\otimes\Pi_{pn}\otimes I_q\right)\left(I_{mn}\otimes\mathrm{vec}(B^T)\right)\Pi_{mn}\\
&-\left(\mathbf{x}^T\otimes(A_{MN}^{\dagger}\otimes B_{PQ}^{\dagger})\right)\left(I_n\otimes\Pi_{qm}\otimes I_p)(I_{mn}\otimes\mathrm{vec}(B)\right)\,,\\
\mathscr{P}=&\left(\mathbf{r}^T\otimes\left((A^TMA)^{-1}\otimes(B^TPB)^{-1}\right)\right)\left(I_m\otimes\Pi_{pn}\otimes I_q\right)\left(\mathrm{vec}(A^T)\otimes I_{pq}\right)\Pi_{pq}\\
&-\left(\mathbf{x}^T\otimes(A_{MN}^{\dagger}\otimes B_{PQ}^{\dagger})\right)\left(I_n\otimes\Pi_{qm}\otimes I_p\right)\left(\mathrm{vec}(A)\otimes I_{pq}\right)\,,\\
\mathbf{r}=&C\left(\mathbf{c}-(A\otimes B)\mathbf{x}\right)\,.
\end{aligned}$$

Proof. From Lemma 3.3, we know that $(A\otimes B)^{\dagger}_{CD}$ is continuous and differentiable, and so is $\varphi(\mathbf{a},\mathbf{b},\mathbf{c})=(A\otimes B)^{\dagger}_{CD}\mathbf{c}$. According to (4.4), we obtain

$$\begin{aligned} d\mathbf{x} =& d\left((A\otimes B)^{\dagger}_{CD}\mathbf{c}\right)\\ =& -\left(A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ}\right)(dA\otimes B+A\otimes dB)\mathbf{x}\\ &+\left((A^TMA)^{-1}\otimes(B^TPB)^{-1}\right)\left(dA^T\otimes B^T+A^T\otimes dB^T\right)\mathbf{r}\\ &+\left(A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ}\right)d\mathbf{c}\,. \end{aligned} \tag{4.6}$$

Vectorizing both sides of (4.6) yields

$$\begin{aligned} d\mathbf{x} =& \mathrm{vec}(d\mathbf{x})\\ =& \mathrm{vec}\left(-(A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ})(dA\otimes B+A\otimes dB)\mathbf{x}\right)\\ &+\mathrm{vec}\left(\left((A^TMA)^{-1}\otimes(B^TPB)^{-1}\right)\left(dA^T\otimes B^T+A^T\otimes dB^T\right)\mathbf{r}\right)+\mathrm{vec}\left((A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ})d\mathbf{c}\right)\\ =& -\left(\mathbf{x}^T\otimes(A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ})\right)\tilde{\Pi}\left[\mathrm{vec}(dA)\otimes\mathrm{vec}(B)+\mathrm{vec}(A)\otimes\mathrm{vec}(dB)\right]\\ &+\left(\mathbf{r}^T\otimes\left((A^TMA)^{-1}\otimes(B^TPB)^{-1}\right)\right)\tilde{G}\left[\mathrm{vec}(dA^T)\otimes\mathrm{vec}(B^T)+\mathrm{vec}(A^T)\otimes\mathrm{vec}(dB^T)\right]\\ &+(A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ})d\mathbf{c}\\ =& -\left(\mathbf{x}^T\otimes(A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ})\right)\tilde{\Pi}\left[\left(I_{mn}\otimes\mathrm{vec}(B)\right)\mathrm{vec}(dA)+\left(\mathrm{vec}(A)\otimes I_{pq}\right)\mathrm{vec}(dB)\right]\\ &+\left(\mathbf{r}^T\otimes\left((A^TMA)^{-1}\otimes(B^TPB)^{-1}\right)\right)\tilde{G}\left[\left(I_{mn}\otimes\mathrm{vec}(B^T)\right)\Pi_{mn}\mathrm{vec}(dA)\right.\\ &\left.+\left(\mathrm{vec}(A^T)\otimes I_{pq}\right)\Pi_{pq}\mathrm{vec}(dB)\right]+(A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ})d\mathbf{c}\,, \end{aligned}$$

where

$$\begin{aligned} \tilde{\Pi} &= \left(I_n\otimes\Pi_{qm}\otimes I_p\right),\\ \tilde{G} &= \left(I_m\otimes\Pi_{pn}\otimes I_q\right)\ , \end{aligned}$$

or

$$d\mathbf{x}=\left[\mathscr{Q}\ \mathscr{P}\ A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ}\right]\left[d\mathbf{a}^T\ d\mathbf{b}^T\ d\mathbf{c}^T\right]^T.$$

Thus the Fréchet derivative of φ is given by

$$\varphi'(\mathbf{a},\mathbf{b},\mathbf{c})=\left[\mathscr{Q}\ \mathscr{P}\ A^{\dagger}_{MN}\otimes B^{\dagger}_{PQ}\right]\ .$$

From the Fréchet derivative of $\varphi(\mathbf{a},\mathbf{b},\mathbf{c})$, we can obtain the explicit expressions of the norm-wise, mixed, and component-wise condition number for the KPWLS problem.

Corollary 4.1. *Under the same assumptions as in Lemma* 4.1*, we have*

$$\kappa^{wls}(A\otimes B,\mathbf{c}):=\lim_{\epsilon\to 0}\sup_{\substack{\sqrt{\|\Delta A\|_F^2+\|\Delta B\|_F^2+\|\Delta\mathbf{c}\|_2^2}\\ \leqslant\epsilon\sqrt{\|A\|_F^2+\|B\|_F^2+\|\mathbf{c}\|_2^2}}}\frac{\|\Delta\mathbf{x}\|_2}{\epsilon\|\mathbf{x}\|_2}$$
$$=\frac{\left\|[\mathscr{Q}\ \mathscr{P}\ A_{MN}^{\dagger}\otimes B_{PQ}^{\dagger}]\right\|_2\sqrt{\|A\|_F^2+\|B\|_F^2+\|\mathbf{c}\|_2^2}}{\|\mathbf{x}\|_2},$$

$$m^{wls}(A\otimes B,\mathbf{c}):=\lim_{\epsilon\to 0}\sup_{\substack{|\Delta A|\leqslant\epsilon|A|\\ |\Delta B|\leqslant\epsilon|B|\\ |\Delta\mathbf{c}|\leqslant\epsilon|\mathbf{c}|}}\frac{1}{\epsilon}\frac{\|\Delta\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty}$$
$$=\frac{\left\||\mathscr{Q}|\mathrm{vec}(|A|)+|\mathscr{P}|\mathrm{vec}(|B|)+(|A_{MN}^{\dagger}|\otimes|B_{PQ}^{\dagger}|)|\mathbf{c}|\right\|_\infty}{\|\mathbf{x}\|_\infty},$$

$$c^{wls}(A\otimes B,\mathbf{c}):=\lim_{\epsilon\to 0}\sup_{\substack{|\Delta A|\leqslant\epsilon|A|\\ |\Delta B|\leqslant\epsilon|B|\\ |\Delta\mathbf{c}|\leqslant\epsilon|\mathbf{c}|}}\frac{1}{\epsilon}\left\|\frac{\Delta\mathbf{x}}{\mathbf{x}}\right\|_\infty$$
$$=\left\|\frac{|\mathscr{Q}|\mathrm{vec}(|A|)+|\mathscr{P}|\mathrm{vec}(|B|)+(|A_{MN}^{\dagger}|\otimes|B_{PQ}^{\dagger}|)|\mathbf{c}|}{\mathbf{x}}\right\|_\infty.$$

The proof of this corollary is similar to that of Theorem 3.1, so we omit it.

The following corollary gives the computable upper bounds for these three condition numbers.

Corollary 4.2. *Under the same assumptions as in Lemma* 4.1*, we have*

$$\kappa^{wls}(A\otimes B,\mathbf{c})\leqslant\frac{\widetilde{K}\sqrt{\|A\|_F^2+\|B\|_F^2+\|\mathbf{c}\|_2^2}}{\|\mathbf{x}\|_2}+\frac{\left\|A_{MN}^{\dagger}\otimes B_{PQ}^{\dagger}\right\|_2\sqrt{\|A\|_F^2+\|B\|_F^2+\|\mathbf{c}\|_2^2}}{\|\mathbf{x}\|_2}$$
$$:=\kappa^{wls}(A\otimes B,\mathbf{c})^{upper},$$

$$m^{wls}(A\otimes B,\mathbf{c})\leqslant\frac{2\left\||(A_{MN}^{\dagger}\otimes B_{PQ}^{\dagger})|(|A|\otimes|B|)|\mathbf{x}|\right\|_\infty}{\|\mathbf{x}\|_\infty}$$
$$+\frac{2\left\||(A^TMA)^{-1}\otimes(B^TNB)^{-1}|(|A|^T\otimes|B|^T)|\mathbf{r}|\right\|_\infty}{\|\mathbf{x}\|_\infty}+\frac{\left\||A_{MN}^{\dagger}\otimes B_{PQ}^{\dagger}||\mathbf{c}|\right\|_\infty}{\|\mathbf{x}\|_\infty}$$
$$:=m^{wls}(A\otimes B,\mathbf{c})^{upper},$$

$$c^{wls}(A\otimes B,\mathbf{c})\leqslant 2\left\||D_{\mathbf{x}}^{\ddagger}||(A_M^{\dagger}\otimes B_N^{\dagger})|(|A|\otimes|B|)|\mathbf{x}|\right\|_\infty$$
$$+2\left\||D_{\mathbf{x}}^{\ddagger}||(A^TMA)^{-1}\otimes(B^TNB)^{-1}|(|A|^T\otimes|B|^T)|\mathbf{r}|\right\|_\infty+\left\||D_{\mathbf{x}}^{\ddagger}||A_M^{\dagger}\otimes B_N^{\dagger}||\mathbf{c}|\right\|_\infty$$
$$:=c^{wls}(A\otimes B,\mathbf{c})^{upper},$$

where

$$\widetilde{K} = \left(\|\mathbf{r}\|_2\|(A^TMA)^{-1}\|_2\|(B^TPB)^{-1}\|_2 + \|\mathbf{x}\|_2\|A^{\dagger}_{MN}\|_2\|B^{\dagger}_{PQ}\|_2\right)\left(\|B\|_F + \|A\|_F\right) .$$

Proof. From Lemma 2.2,

$$\begin{aligned}\|\mathscr{Q}\|_2 \leqslant & \left\|(\mathbf{r}^T \otimes \left((A^TMA)^{-1} \otimes (B^TPB)^{-1}\right))(I_m \otimes \Pi_{pn} \otimes I_q)\left(I_{mn} \otimes \mathrm{vec}(B^T)\right)\Pi_{mn}\right\|_2 \\ & + \left\|-(\mathbf{x}^T \otimes (A^{\dagger}_{MN} \otimes B^{\dagger}_{PQ}))(I_n \otimes \Pi_{qm} \otimes I_p)\left(I_{mn} \otimes \mathrm{vec}(B)\right)\right\|_2 \\ \leqslant & \|r\|_2 \left\|(A^TMA)^{-1}\right\|_2 \left\|(B^TPB)^{-1}\right\|_2 \left\|\mathrm{vec}(B^T)\right\|_2 + \|\mathbf{x}\|_2 \left\|A^{\dagger}_{MN}\right\|_2 \left\|B^{\dagger}_{PQ}\right\|_2 \|\mathrm{vec}(B)\|_2 \\ = & \left(\|\mathbf{r}\|_2 \left\|(A^TMA)^{-1}\right\|_2 \left\|(B^TPB)^{-1}\right\|_2 + \|\mathbf{x}\|_2 \left\|A^{\dagger}_{MN}\right\|_2 \left\|B^{\dagger}_{PQ}\right\|_2\right) \|B\|_F\end{aligned}$$

and

$$\|\mathscr{P}\|_2 \leqslant \left(\|\mathbf{r}\|_2 \left\|(A^TMA)^{-1}\right\|_2 \left\|(B^TPB)^{-1}\right\|_2 + \|\mathbf{x}\|_2 \left\|A^{\dagger}_{MN}\right\|_2 \left\|B^{\dagger}_{PQ}\right\|_2\right) \|A\|_F .$$

Furthermore, we have

$$\begin{aligned}\left\|\left[\mathscr{Q}\ \mathscr{P}\ A^{\dagger}_{MN} \otimes B^{\dagger}_{PQ}\right]\right\|_2 \leqslant & \|\mathscr{Q}\|_2 + \|\mathscr{P}\|_2 + \left\|A^{\dagger}_{MN} \otimes B^{\dagger}_{PQ}\right\|_2 \\ \leqslant & \widetilde{K} + \left\|A^{\dagger}_{MN} \otimes B^{\dagger}_{PQ}\right\|_2 .\end{aligned}$$

According to Corollary 4.2, we can obtain the upper bounds $\kappa^{wls}(A \otimes B, c)^{upper}$.

From Lemma 2.2, we obtain

$$\begin{aligned}|\mathscr{Q}||\mathrm{vec}(A)| \leqslant & \left|\left(\mathbf{r}^T \otimes ((A^TMA)^{-1} \otimes (B^TPB)^{-1})\right)\tilde{G}(I_{mn} \otimes \mathrm{vec}(B^T))\,\Pi_{mn}\right| |\mathrm{vec}(A)| \\ & + \left|(\mathbf{x}^T \otimes (A^{\dagger}_{MN} \otimes B^{\dagger}_{PQ}))\tilde{\Pi}(I_{mn} \otimes \mathrm{vec}(B))\right| |\mathrm{vec}(A)| \\ \leqslant & \left[\left(|\mathbf{r}^T| \otimes |(A^TMA)^{-1} \otimes (B^TPB)^{-1}|\right)\tilde{G}\left(I_{mn} \otimes \mathrm{vec}(|B^T|)\right)\right]\left|\mathrm{vec}(|A^T|)\right| \\ & + \left[(|\mathbf{x}^T| \otimes |A^{\dagger}_{MN} \otimes B^{\dagger}_{PQ}|)\tilde{\Pi}(I_{mn} \otimes \mathrm{vec}(|B|))\right]\left|\mathrm{vec}(|A|)\right| \\ = & \left(|\mathbf{r}^T| \otimes |(A^TMA)^{-1} \otimes (B^TPB)^{-1}|\right)\tilde{G}\mathrm{vec}\left(\mathrm{vec}(|B^T|)\mathrm{vec}(|A^T|)^T\right) \\ & + \left(|\mathbf{x}^T| \otimes |A^{\dagger}_{MN} \otimes B^{\dagger}_{PQ}|\right)\tilde{\Pi}\mathrm{vec}\left(\mathrm{vec}(|B|)\mathrm{vec}(|A|)^T\right) \\ = & \left(|\mathbf{r}^T| \otimes |(A^TMA)^{-1} \otimes (B^TPB)^{-1}|\right)\mathrm{vec}\left(|A^T| \otimes |B^T|\right) \\ & + \left(|\mathbf{x}^T| \otimes |A^{\dagger}_{MN} \otimes B^{\dagger}_{PQ}|\right)\mathrm{vec}\left(|A| \otimes |B|\right) \\ = & \left|A^{\dagger}_{MN} \otimes B^{\dagger}_{PQ}\right|\left(|A| \otimes |B|\right)|\mathbf{x}| + \left|(A^TMA)^{-1} \otimes (B^TPB)^{-1}\right|\left(|A|^T \otimes |B|^T\right)|\mathbf{r}| ,\end{aligned}$$

where

$$\tilde{\Pi} = (I_n \otimes \Pi_{qm} \otimes I_p), \qquad \tilde{G} = (I_m \otimes \Pi_{pn} \otimes I_q) .$$

Similarly,

$$|\mathscr{P}||\mathrm{vec}(B)| \leqslant \left|A^{\dagger}_{MN} \otimes B^{\dagger}_{PQ}\right| \left(|A| \otimes |B|\right)|\mathbf{x}| + \left|(A^T MA)^{-1} \otimes (B^T PB)^{-1}\right| \left(|A|^T \otimes |B|^T\right)|\mathbf{r}| \,.$$

From Corollary 4.2 and the matrix norm triangular inequality, we can obtain the upper bounds $M^{wls}(A \otimes B, \mathbf{c})^{upper}$ and $c^{wls}(A \otimes B, \mathbf{c})^{upper}$. □

5. Numerical Example

In this section, we provide the examples for illustration. All computations were performed using MATLAB 7.0. The relative machine precision was 2.2×10^{-16}.

Positive definite matrices M, N, P and Q were taken randomly, and matrices A and B generated randomly such that $\mathrm{rank}(A \otimes B) = nq$.

Table 1 shows that upper bounds that are about the same magnitude as their exact condition numbers. Table 2 shows the upper bound of norm-wise condition numbers that are around two orders of magnitude larger than their exact counterparts, while those of mixed and component-wise condition numbers are about one order of magnitude larger than their exact values.

For further illustration, positive definite matrices $M, N, P, Q \in \mathbb{R}^{8\times 8}$ and full column rank matrices A and $B \in \mathbb{R}^{8\times 8}$ were generated randomly in 100 runs.

The numerical results are shown in Fig. 1, where κ, m, c, κ_w, m_w and c_w are equal to $\kappa((A\otimes B)^{\dagger}_{CD})$, $m((A\otimes B)^{\dagger}_{CD})$, $c((A\otimes B)^{\dagger}_{CD})$, $\kappa^{wls}(A\otimes B, \mathbf{c})$, $m^{wls}(A\otimes B, \mathbf{c})$ and $c^{wls}(A\otimes B, \mathbf{c})$, respectively. Also, κ^{upper}, m^{upper}, c^{upper}, κ_w^{upper}, m_w^{upper} and c_w^{upper} are equal to $\kappa((A \otimes$

Table 1: Weighted Moore-Penrose inverse of a Kronecker product.

	$p=m=4, q=n=3$	$p=m=8, q=n=5$	$p=m=12, q=n=6$
$\kappa((A\otimes B)^{\dagger}_{CD})$	1.3921e+001	2.2485e+001	3.4869e+001
$\kappa((A\otimes B)^{\dagger}_{CD})^{upper}$	4.9667e+001	6.2037e+001	7.1191e+001
$m((A\otimes B)^{\dagger}_{CD})$	9.5813	1.6362e+001	2.6815e+001
$m((A\otimes B)^{\dagger}_{CD})^{upper}$	2.2748e+001	2.6637e+001	3.8591e+001
$c((A\otimes B)^{\dagger}_{CD})$	1.6858e+002	5.9543e+001	4.3705e+003
$c((A\otimes B)^{\dagger}_{CD})^{upper}$	2.1159e+002	6.4579e+002	4.6430e+003

Table 2: Weighted linear least squares problems involving Kronecker products.

	$p=m=4, q=n=3$	$p=m=8, q=n=5$	$p=m=12, q=n=6$
$\kappa^{wls}(A\otimes B, \mathbf{c})$	3.2863e+001	3.3469e+001	6.8390e+001
$\kappa^{wls}(A\otimes B, \mathbf{c})^{upper}$	2.0199e+003	6.7526e+003	1.8640e+004
$m^{wls}(A\otimes B, \mathbf{c})$	2.399e+001	2.3796e+001	4.3432e+001
$m^{wls}(A\otimes B, \mathbf{c})^{upper}$	2.4765e+002	6.5998e+002	1.3813e+003
$c^{wls}(A\otimes B, \mathbf{c})$	2.1002e+002	9.3953e+002	1.9027e+004
$c^{wls}(A\otimes B, \mathbf{c})^{upper}$	1.5914e+003	2.1122e+004	5.5902e+005

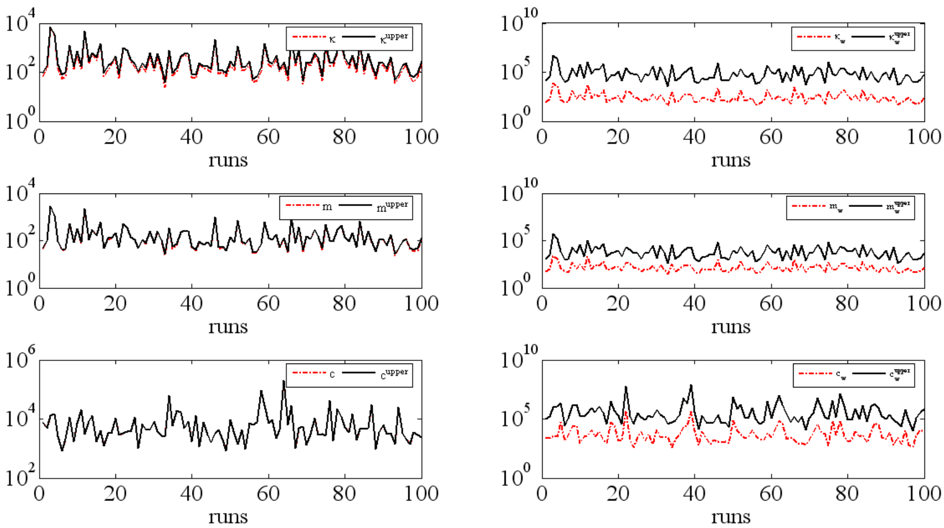

Figure 1: Condition numbers and their upper bounds.

$B)^{\dagger}_{CD})^{upper}$, $m((A\otimes B)^{\dagger}_{CD})^{upper}$, $c((A\otimes B)^{\dagger}_{CD})^{upper}$, $\kappa^{wls}(A\otimes B,\mathbf{c})^{upper}$, $m^{wls}(A\otimes B,\mathbf{c})^{upper}$ and $c^{wls}(A\otimes B,\mathbf{c})^{upper}$, respectively.

From Fig. 1 we see that sometimes the proposed upper bounds can provide rough estimates for the condition numbers.

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