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Elementwise Minimal Nonnegative Solutions for a Class of Nonlinear Matrix Equations

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Abstract. The existence of elementwise minimal nonnegative solutions of the nonlinear matrix equations

$$A^T X^2 A - X + I = 0,$$
$$A^T X^n A - X + I = 0, \quad n > 2$$

are studied. Using Newton's method with the zero initial guess, we show that under suitable conditions the corresponding iterations monotonically converge to the elementwise minimal nonnegative solutions of the above equations. Numerical experiments confirm theoretical results and the efficiency of the method.

AMS subject classifications: 15A24, 65F30

Key words: Elementwise minimal nonnegative solution, Newton's method, monotonic convergence, nonlinear matrix equation.

1. Introduction

This study deals with the nonlinear matrix equations

$$F(X) = A^T X^2 A - X + I = 0, \tag{1.1}$$
$$G(X) = A^T X^n A - X + I = 0, \quad n > 2, \tag{1.2}$$

where A and X are, respectively, given and unknown real square matrices of the same size and I is the unit matrix. The most popular general form of (1.1) and (1.2) is

$$X^s + A^T \mathscr{F}(X) A = Q, \quad Q > 0. \tag{1.3}$$

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In the past few decades, the Eq. (1.3) has been extensively studied. In particular, the following special cases of the Eq. (1.3) have been considered:

- $s = 1$ and $\mathscr{F}(X) = X^{-1}$ — i.e. a symmetric nonlinear equation [1,8,11,15,16].
- $s = 1$ and $\mathscr{F}(X) = (X + B)^{-1}$ — i.e. a discrete algebraic Riccati equation [6,13].
- $s = 2$ and $\mathscr{F}(X) = X$ — i.e. a quadratic matrix equation [2,10,19].
- $s = 1$ and $\mathscr{F}(X) = X^{-2}$ — i.e. a symmetric nonlinear matrix equation [5,22].

In addition, Liao and Zhang [14] studied preconditioned iteration methods for a class of complex linear algebraic systems

$$(W + iT)u = c$$

for symmetric matrices W and T, one of which is nonsingular. Chacha and Naqvi [4] investigated mixed and condition numbers of the nonlinear matrix equation

$$X^p - A^* e^X A = I \tag{1.4}$$

and proposed a fixed point method for the Eq. (1.4).

The Eqs. (1.1) and (1.2) are similar to the equation

$$X + A^* X^q A = I,$$

positive definite solutions of which for $q = 0.5, 0.1, 1.2$ have been derived by Zhang *et al.* [21] by iterative methods. Ran and Reurings [17] obtained general results for the matrix equation $X + A^*\mathfrak{F}(X)A = Q$, where $\mathfrak{F}$ is a map from the set of all semidefinite matrices into $\mathbb{C}^{n\times n}$ satisfying monotonicity properties.

The corresponding studies are mainly restricted to positive definite or Hermitian positive definite solutions. Seo *et al.* [19] examined elementwise minimal nonnegative solutions of the equation $Q(X) = AX^2 + BX + C = 0$, where A and C are nonnegative matrices and $-B$ is a nonsingular M-matrix and evaluated them by the Newton's method. Moreover, Seo and Kim [18] proposed a relaxed Newton's method with the zero starting point for a matrix polynomial equation with positive coefficients arising in stochastic models and proved the existence of an elementwise nonnegative solution.

It is worth noting that the Newton's method is mainly used to study elementwise minimal nonnegative solutions of matrix equations, since with zero initial approximation the method converges monotonically to the solution mentioned — cf. [7,12]. On the other hand, Newton's method starting with a positive definite initial matrix may converge to a positive definite solution [9]. Nevertheless, to the best of author's knowledge, in general case the elementwise nonnegative solutions of the Eqs. (1.2) and (1.1) have not been studied and this work aims at the elementwise nonnegative solutions of such equations.

The following notations are used throughout this paper: $\rho(A)$ denotes the spectral radius of the matrix A, A^T the transpose of A, 0 a square null matrix, $A \otimes B = [a_{ij}B]$ the Kronecker product of matrices A and B, whereas a_i is the i^{th} column of matrix A and $\text{vec}(A) = [a_1^T, a_2^T, \dots, a_n^T]^T$ the column-wise vector representation of A.