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A Generalised CRI Iteration Method for Complex Symmetric Linear Systems

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Abstract. A generalisation of the combination method of real and imaginary parts for complex symmetric linear systems based on the introduction of an additional parameter is proposed. Sufficient conditions for the convergence of the method are derived. Numerical examples show the efficiency of this algorithm.

AMS subject classifications: 15A24, 65F10

Key words: Complex linear system, iterative method, convergence, optimal parameters.

1. Introduction

Let $\mathrm{i} = \sqrt{-1}$ be the imaginary unit, $p, q \in \mathbb{R}^n$ and $b = p + \mathrm{i}q \in \mathbb{C}^n$. We consider the system of linear equations

$$Ax = b, \quad x = y + \mathrm{i}z, \quad y, z \in \mathbb{R}^n \tag{1.1}$$

with a nonsingular complex symmetric matrix $A \in \mathbb{C}^{n\times n}$ of the form

$$A = W + \mathrm{i}T.$$

Assume that $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semidefinite matrices. Complex symmetric systems arise in various applications, including wave propagation [34], FFT-based solutions of time-dependent PDEs [11], diffuse optical tomography [1], numerical methods for time-dependent Schrödinger equation [15], Maxwell's equations [23], molecular scattering [31], structural dynamics [25], modelling of electrical power systems [24], lattice quantum chromodynamics [27], quantum chemistry, eddy current problems [3], discretisation of self-adjoint integro-differential equations of environmental modelling [28], and so on [2, 4, 6, 9, 14].

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In recent years, the solution of matrix equations attracts more and more interest [12, 16–20, 26, 32, 35]. Iterative algorithms are a powerful and very successful technique in linear systems and system identification. Various iteration methods have been used for solving the complex linear system (1.1) — e.g. the conjugate orthogonal conjugate gradient method, the complex symmetric method and quasi-minimal residual method [12, 26, 35]. These methods are directly applicable to the complex linear system (1.1). However, one can avoid using complex arithmetic by rewriting complex systems of linear equations as 2×2 block real linear systems [2, 10, 13, 39]. Instead of solving the original complex linear system (1.1), Salkuyeh *et al.* [33] applied the generalised successive overrelaxation (GSOR) iterative method to an equivalent real system. In addition, Hezari *et al.* [21] presented a preconditioned GSOR iterative method and studied conditions when the spectral radius of the iteration matrix of the preconditioned GSOR method is smaller than that of the GSOR method, also finding optimal iteration parameters.

Since W and T in (1.1) are real symmetric matrices, the Hermitian H and skew-Hermitian S parts of the complex symmetric matrix A are

$$H = \frac{1}{2}\left(A + A^H\right) = W \quad \text{and} \quad S = \frac{1}{2}\left(A - A^H\right) = \mathrm{i}T.$$

Using the special structure of the complex matrix $A \in \mathbb{C}^{n\times n}$ and the Hermitian and skew-Hermitian splitting (HSS) method [7], Bai *et al.* [4] developed a modified HSS (MHSS) iteration method, which converges unconditionally to the unique solution of the Eq. (1.1). Besides, in order to accelerate the convergence rate of the MHSS method, Bai *et al.* [5] proposed the following preconditioned MHSS (PMHSS) method.

PMHSS Iteration Method. Let $x^{(0)} \in \mathbb{C}^n$ be an arbitrary initial guess. For $k = 0, 1, 2, \ldots$, until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty} \subset \mathbb{C}^n$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{aligned} (\alpha V + W)x^{(k+1/2)} &= (\alpha V - \mathrm{i}T)x^{(k)} + b, \\ (\alpha V + T)x^{(k+1)} &= (\alpha V + \mathrm{i}W)x^{(k+1/2)} - \mathrm{i}b, \end{aligned} \tag{1.2}$$

where α is a given positive constant and $V \in \mathbb{R}^{n\times n}$ is a prescribed symmetric positive definite matrix.

If V is the identity matrix, then the PMHSS method becomes the MHSS method. Bai *et al.* [5] proved that for any initial guess, the PMHSS iteration method converges to the unique solution of (1.1). Hezari *et al.* [22] presented a new stationary matrix splitting iteration method, called Scale-Splitting, to solve the complex system (1.1). Convergence theory and spectral properties of the corresponding preconditioned matrix have been also established. Zheng *et al.* [40] exploited the symmetry of the PMHSS method and employed scaling technique to reconstruct complex linear system (1.1). Moreover, it was shown that the double-step scale splitting iteration method proposed in [40] is unconditionally convergent and converges faster than the PMHSS method. Liao and Zhang [30] introduced a block multiplicative preconditioner and the corresponding block multiplicative iteration method for complex symmetric linear algebraic systems.