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A Local Positive (Semi)Definite Shift-Splitting Preconditioner for Saddle Point Problems with Applications to Time-Harmonic Eddy Current Models

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Abstract. A local positive (semi)definite shift-splitting preconditioner for non-Hermitian saddle point problems arising in finite element discretisations of hybrid formulations of time-harmonic eddy current models is constructed. The convergence of the corresponding iteration methods is proved and the spectral properties of the associated preconditioned saddle point matrices are studied. Numerical experiments show the efficiency of the proposed preconditioner for Krylov subspace methods.

AMS subject classifications: 65F10, 65F50

Key words: Saddle point problem, splitting iteration, preconditioning, convergence, time-harmonic eddy current model.

1. Introduction

Let $A \in \mathbb{C}^{n\times n}$ be a non-Hermitian positive (semi)definite matrix — i.e. $H = (1/2)(A+A^*)$ is positive (semi)definite, and $B \in \mathbb{C}^{m\times n}$ with $m \le n$ be a full rank matrix. We consider iterative solutions of the following large sparse saddle point problem:

$$\mathscr{A} w \equiv \begin{bmatrix} A & B^* \\ -B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \equiv p, \tag{1.1}$$

where B^* denotes the conjugate transpose of B and $f \in \mathbb{C}^n$, $g \in \mathbb{C}^m$ are given vectors. It is well-known that if A and B are, respectively, positive definite and full rank matrices or if

$$\mathrm{null}(A) \cap \mathrm{null}(B) = \{0\}, \quad \mathrm{null}(B^*) = \{0\},$$

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then the non-Hermitian saddle point matrix $\mathscr{A}$ is nonsingular [6,17]. The saddle point problem (1.1) appears in various applications, including computational fluid dynamics [30], constrained and weighted least squares problems [34], electromagnetism [19], time-harmonic eddy current models [5,36,38,39], geomechanics [18], and meshfree discretisation of elasticity problems [15,27]. The reader can also consult [17] and the references therein for more information about the problem.

Recently, a lot of efforts have been spent on iteration methods for the problem (1.1). The list of the methods studied includes classical Uzawa iteration method [2] and its generalisations [13,20,28,31], Hermitian and skew-Hermitian splitting (HSS) iteration methods [11] and its variants [9,10,26,33,45], shift-splitting iteration methods [1,14,22–24,29,41,42], residual reduction algorithms [3] and Krylov subspace iteration methods [40]. If $\mathscr{A}$ is a non-Hermitian and/or ill-conditioned matrix, the preconditioning is often used to accelerate the convergence. For example, block diagonal and block triangular preconditioners are considered in [4,12], HSS preconditioners in [16,21,25] and shift-splitting preconditioners in [23,43] — cf. also [17,35] and references therein. These iteration methods and preconditioners often depend on the problem studied and have to be adjusted with respect to the corresponding coefficient matrices.

In this work, we focus on a class of non-Hermitian saddle point problems arising in the finite element discretisations of hybrid formulations of time-harmonic eddy current models [38,39]. A model often used to simulate the electromagnetic phenomena of alternating currents at low frequencies can be described by the equations

$$\begin{aligned}
&\mathbf{curl}\big(\sigma^{-1}\mathbf{curl}\,\mathbf{H}_C\big)+i\omega\mu\mathbf{H}_C=\mathbf{curl}\big(\sigma^{-1}\mathbf{J}_{e,C}\big) && \text{in} \quad \Omega_C,\\
&\mathbf{curl}\big(\mu^{-1}\mathbf{curl}\,\mathbf{E}_I\big)=-i\omega\mathbf{J}_{e,I} && \text{in} \quad \Omega_I,\\
&\mathrm{div}(\epsilon\mathbf{E}_I)=0 && \text{in} \quad \Omega_I,\\
&\mu^{-1}\mathbf{curl}\,\mathbf{E}_I\times\mathbf{n}=0 && \text{on} \quad \partial\Omega,\\
&\epsilon\mathbf{E}_I\cdot\mathbf{n}=0 && \text{on} \quad \partial\Omega,\\
&\mathbf{H}_C\times\mathbf{n}=(-i\omega\mu)^{-1}\mathbf{curl}\,\mathbf{E}_I\times\mathbf{n} && \text{on} \quad \Gamma,\\
&\mathbf{E}_I\times\mathbf{n}=\sigma^{-1}\big(\mathbf{curl}\,\mathbf{H}_C-\mathbf{J}_{e,C}\big)\times\mathbf{n} && \text{on} \quad \Gamma,
\end{aligned} \tag{1.2}$$

where $\mathbf{E}$, $\mathbf{H}$, $\mathbf{J}_e$, μ, σ, ω and i, respectively, denote electric field, magnetic field, generator current, magnetic permeability, electric conductivity, a nonzero angular frequency and the imaginary unit. The computational domain $\Omega\subset\mathbb{R}^3$ is a simply connected Lipschitz polyhedron, which consists of the conducting region $\Omega_C\subset\Omega$ and its complement $\Omega_I=\Omega\backslash\bar{\Omega}_C$. We assume that Ω_C and Ω_I are Lipschitz polyhedrons, Ω_C is connected but not necessarily simply connected and by $\bar{\Omega}_C$ and $\bar{\Omega}_I$ we denote the closures of Ω_C and Ω_I, respectively. Applying the finite element method of [39] to (1.2), one obtains the following linear system:

$$\begin{bmatrix} M_C-iS_C & -iD^T & B_C^T & 0\\ -iD & S_I+\tau B_I^TB_I & 0 & B_I^T\\ -B_C & 0 & 0 & 0\\ 0 & -B_I & 0 & 0\end{bmatrix}\begin{bmatrix} H_C\\ \widetilde{E}_I\\ \widetilde{Q}\\ \widetilde{\Phi}_I\end{bmatrix}=\begin{bmatrix}-iF_C\\ -iG_I\\ 0\\ 0\end{bmatrix}, \tag{1.3}$$