East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.050319.110619

Vol. **10**, No. 1, pp. 106-122
February 2020

Propagation Property and Application to Inverse Scattering for Fractional Powers of Negative Laplacian

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Received 5 March 2019; Accepted (in revised version) 11 June 2019.

Abstract. The propagation estimate for the usual free Schrödinger operator established by Enss in 1983, was successfully used by Enss and Weder in inverse scattering in 1995. This approach has been called the Enss-Weder time-dependent method. We derive the same type of estimate but for fractional powers of the negative Laplacian and apply it in inverse scattering. It is found that the high-velocity limit of the scattering operator uniquely determines the short-range interactions.

AMS subject classifications: 81Q10, 81U05, 81U40

Key words: Scattering theory, inverse problem, fractional Laplacian.

1. Introduction

Let D_x denote the differential operator $-i\nabla_x = -i(\partial_{x_1}, \dots, \partial_{x_n})$. The fractional power of the negative Laplacian acting on the space $L^2(\mathbb{R}^n)$ is the operator

$$H_{0,\rho} = \omega_\rho(D_x), \quad \frac{1}{2} \leqslant \rho \leqslant 1,$$

defined by the Fourier multiplier with the symbol

$$\omega_\rho(\xi) = \frac{|\xi|^{2\rho}}{2\rho}.$$

More precisely, $H_{0,\rho}$ is the Fourier integral operator

$$H_{0,\rho}\phi(x) = (\mathscr{F}^*\omega_\rho(\xi)\mathscr{F}\phi)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}\omega_\rho(\xi)\phi(y)dyd\xi,$$

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where ϕ belongs to the Sobolev space $H^{2\rho}(\mathbb{R}^n)$. In particular, $H_{0,1}$ is the free Schrödinger operator

$$\omega_1(D_x) = -\frac{\Delta_x}{2} = -\frac{1}{2}\sum_{j=1}^{n}\partial_{x_j}^2,$$

and $H_{0,1/2}$ the massless relativistic Schrödinger operator $\omega_{1/2}(D_x) = \sqrt{-\Delta_x}$.

Let $F(X)$ refer to the usual characteristic function of the set X and let $\chi \in C^\infty(\mathbb{R}^n)$ be a function such that

$$\chi(x) = \begin{cases} 1, & |x| \geqslant 2, \\ 0, & |x| \leqslant 1. \end{cases}$$

In Section 2, we prove the following Enss-type propagation estimate for $e^{-itH_{0,\rho}}$.

Theorem 1.1. *Let $f \in C_0^\infty(\mathbb{R}^n)$ and* $\operatorname{supp} f \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant \eta\}$ *for a given $\eta > 0$. Choose $v \in \mathbb{R}^n$ such that $|v| > \eta$ and*

$$\begin{aligned} &16n(1-\rho)(|v|-\eta)^{2\rho-2}\eta \leqslant |v|^{2\rho-1}, \quad \frac{1}{2} \leqslant \rho < 1, \\ &8\eta \leqslant |v|, \quad \rho = 1. \end{aligned} \tag{1.1}$$

For $t \in \mathbb{R}$ and $N \in \mathbb{N}$, the inequality

$$\left\| \chi\left(\frac{x-(\nabla_\xi \omega_\rho)(v)t}{|v|^{2\rho-1}|t|/4}\right) e^{-itH_{0,\rho}} f(D_x - v) F\left(|x| \leqslant \frac{|v|^{2\rho-1}|t|}{16}\right)\right\| \leqslant C_N\left(1+|v|^{2\rho-1}|t|\right)^{-N} \tag{1.2}$$

holds, where $\|\cdot\|$ is the operator norm on $L^2(\mathbb{R}^n)$ and constant $C_N > 0$ depends on n, N and the shape of f.

Let us recall that Enss [5] established the following estimate for the free Schrödinger operator $H_{0,1}$:

$$\left\| F\left(|x-vt| \geqslant \frac{|v||t|}{4}\right) e^{-itD_x^2/2} f(D_x - v) F\left(|x| \leqslant \frac{|v||t|}{16}\right)\right\| \leqslant C_N(1+|v||t|)^{-N}, \tag{1.3}$$

and this estimate is valid not only for spheres, but also for general measurable subsets of $\mathbb{R}^n$ — cf. [5, Proposition 2.10]. Let us briefly discuss the substance of the estimate (1.3). In classical mechanics, D_x represents the momentum or, equivalently, the velocity of the particle of unit mass. In the left-hand side of (1.3), D_x is localised to the neighborhood of v by a cut-off function f. Therefore, during the time evolution of the propagator $e^{-itD_x^2/2}$, the position of the particle changes as

$$x \sim D_x t \sim vt.$$

Since the points on the sphere behave similarly, the center of the sphere moves toward vt from the origin

$$\left\{x \in \mathbb{R}^n \,\middle|\, |x| \leqslant \frac{|v||t|}{16}\right\} \sim \left\{x \in \mathbb{R}^n \,\middle|\, |x-vt| \leqslant \frac{|v||t|}{16}\right\}. \tag{1.4}$$