Numerical Soliton Solutions for a Discrete Sine-Gordon System

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Abstract. In this paper we use an analytical-numerical approach to find, in a systematic way, new 1-soliton solutions for a discrete sine-Gordon system in one spatial dimension. Since the spatial domain is unbounded, the numerical scheme employed to generate these soliton solutions is based on the artificial boundary method. A large selection of numerical examples provides much insight into the possible shapes of these new 1-solitons.

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1 Introduction

The sine-Gordon equation,

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin(u) = 0
\]

(1.1)
is a semilinear hyperbolic equation in 1+1 dimensions. This PDE has its origin in the 19th century where it arose in the study of surfaces of constant negative curvature (cf. [1]). In the second half of the 20th century the sine-Gordon equation has attracted considerable attention, owing to its importance in the mathematical modeling of various physical phenomena, for example in nonlinear optics (propagation of pulses in resonant media); superconductivity (wave propagation in a Josephson transmission line); condensed matter...
physics (charge density waves in periodic pinning potentials); and in solid state physics (propagation of a dislocation in a crystal). Details and additional examples can be found in [2–5].

A very important property of the sine-Gordon equation (1.1) is the existence of soliton solutions. Many of these special solutions have been obtained in closed form, by using analytical methods such as Bäcklund transformations [6], the nonlinear separation of variables method [2]; see also [3] (Chapter 6). The known soliton solutions of (1.1) mainly can be classified as follows:

1. 1-soliton solutions: Two 1-soliton solutions are given by [7, 8]

\[ u(x,t) = 4 \arctan \left( \frac{\pm x - \mu t - x_0}{\sqrt{1 - \mu^2}} \right), \quad \mu^2 < 1, \]  

(1.2)

and

\[ u(x,t) = -\pi + 4 \arctan \left( \frac{\pm x - \mu t - x_0}{\sqrt{\mu^2 - 1}} \right), \quad \mu^2 > 1. \]  

(1.3)

Here, \( x_0, \mu \in \mathbb{R} \) and \( |\mu| \neq 1 \).

2. Breather solutions: Two breather solutions to (1.1) are given by [7, 8]

\[ u(x,t) = 4 \arctan \left( \frac{\mu \sinh(kx + A)}{k \cosh(\mu t + B)} \right), \quad \mu^2 = k^2 + 1, \]  

(1.4)

and

\[ u(x,t) = 4 \arctan \left( \frac{\mu \sin(kx + A)}{k \cosh(\mu t + B)} \right), \quad \mu^2 = 1 - k^2 > 0. \]  

(1.5)

Here, \( A \) and \( B \) are arbitrary (real) constants, and the real numbers \( \mu \) and \( k \) are related by the conditions in (1.4) and (1.5), respectively.

3. N-soliton solutions: An \( N \)-soliton solution for (1.1) is given by

\[ u(x,t) = x + \arccos \left\{ 1 - 2 \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) \ln F(x,t) \right\}, \]  

(1.6)

with

\[ F(x,t) := \det[M_{ij}], \quad M_{ij} := \frac{2}{a_i + a_j} \cosh \left( \frac{z_i + z_j}{2} \right), \]  

\[ z_i := \pm \frac{x - \mu_i t + C_i}{\sqrt{1 - \mu_i^2}}, \quad a_i := \pm \sqrt{\frac{1 - \mu_i^2}{1 + \mu_i^2}}. \]