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A Bilinear Immersed Finite Volume Element Method for the Diffusion Equation with Discontinuous Coefficient†

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Abstract. This paper is to present a finite volume element (FVE) method based on the bilinear immersed finite element (IFE) for solving the boundary value problems of the diffusion equation with a discontinuous coefficient (interface problem). This method possesses the usual FVE method's local conservation property and can use a structured mesh or even the Cartesian mesh to solve a boundary value problem whose coefficient has discontinuity along piecewise smooth nontrivial curves. Numerical examples are provided to demonstrate features of this method. In particular, this method can produce a numerical solution to an interface problem with the usual $\mathcal{O}(h^2)$ (in L^2 norm) and $\mathcal{O}(h)$ (in H^1 norm) convergence rates.

AMS subject classifications: 65N15, 65N30, 65N50, 35R05

Key words: Interface problems, immersed interface, finite volume element, discontinuous coefficient, diffusion equation.

1 Introduction

In many applications, a simulation domain is often formed by several materials separated by curves or surfaces from each other, and this often leads to the so called interface problem consisting of the usual boundary value problem of the diffusion equation, the usual boundary condition, plus jump conditions across the material interface

†This paper is dedicated to Richard E. Ewing on the occasion of his 60th birthday.
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required by pertinent physics. It is well known that efficiently solving this type of interface problem is critical in many applications of engineering and sciences, including flow problems [10, 11, 27, 29, 30, 43, 52], electromagnetic problems [4, 16, 61, 66–70, 78], shape/topology optimization problems [13–15, 29, 36–38, 46, 74], and the modeling of nonlinear phenomena [41, 79, 86], to name just a few. In this paper, we present a finite volume element method with bilinear immersed finite element (IFE) [39, 59] for solving this popular interface problem. This method possesses both the advantages of finite volume element method and those of IFE. In particular, this method can use a Cartesian mesh to solve a boundary value problem with a discontinuous coefficient whose interface consists of nontrivial piecewise smooth curves.

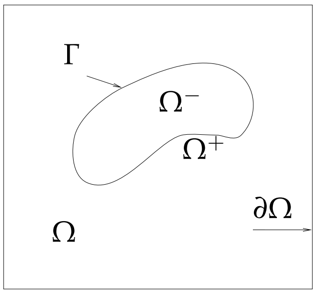

Figure 1: A sketch of the domain for the interface problem.

To be specific, we consider the following boundary value problem:

$$-\nabla\cdot(\beta\nabla u)=f,\quad (x,y)\in\Omega, \tag{1.1}$$

$$u|_{\partial\Omega}=g. \tag{1.2}$$

Here, see the sketch in Fig. 1, without loss of generality, we assume that $\Omega\subset IR^2$ is a rectangular domain, the interface Γ is a curve separating Ω into two sub-domains Ω^-, Ω^+ such that $\overline{\Omega}=\overline{\Omega^-}\cup\overline{\Omega^+}\cup\Gamma$, and the coefficient $\beta(x,y)$ is a piecewise constant function defined by

$$\beta(x,y)=\begin{cases}\beta^-, & (x,y)\in\Omega^-,\\ \beta^+, & (x,y)\in\Omega^+.\end{cases}$$

Because of the discontinuity in the coefficient β, jump conditions are also imposed on the interface Γ:

$$[u]|_\Gamma=0, \tag{1.3}$$

$$\left[\beta\frac{\partial u}{\partial n}\right]|_\Gamma=0. \tag{1.4}$$

Of course, conventional numerical methods can be used to solve interface problem (1.1)-(1.4). Standard discretization techniques such as finite difference (FD), see [73] and references therein, finite volume (FV), see [40] and references therein, and finite element