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Point Sources Identification Problems for Heat Equations

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Abstract. We considered the point source identification problems for heat equations from noisy observation data taken at the minimum number of spatially fixed measurement points. We aim to identify the unknown number of sources and their locations along with their strengths. In our previous work, we proved that minimum measurement points needed under the noise-free setting. In this paper, we extend the proof to cover the noisy cases over a border class of source functions. We show that if the regularization parameter is chosen properly, the problem can be transformed into a poles identification problem. A reconstruction scheme is proposed on the basis of the developed theoretical results. Numerical demonstrations in 2D and 3D conclude the paper.

AMS subject classifications: 35R30 (35K20)

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1 Introduction

Inverse source identification problems are important in many branches of engineering sciences. For examples, an accurate estimation of a pollution source in a river [7], a determination of magnitude of groundwater pollution sources [15] are crucial to environmental protection. Other examples can be found in [20, 21] and the references therein. In general, a complete recovery of the unknown source is not attainable from practically restricted boundary measurements. The inverse source problem becomes solvable if certain *a priori* knowledge is assumed. Inverse problems are in nature *unstable* because the unknown solutions/ parameters have to be determined from indirect observable data

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which contain measurement errors. The major difficulty in establishing any numerical algorithm for approximating the solution is the severe ill-posedness of the problem and the ill-conditioning of the resultant discretized matrix.

The heat conduction process is irreducible in time, while the temperature profile becomes rapidly smoother in time. This means that the characteristic of the solution may not be affected by the observed data. To the knowledge of the authors, the mathematical analysis and efficient algorithms for inverse heat problems are still very limited. For instance, the uniqueness and conditional stability results for heat source identification problem can be found in [3, 4, 26]. Studies on stationary point source problem can be found in [2, 5, 16]. Some reconstruction schemes can be found in [22, 24, 28, 29].

In order to solve an inverse problem of any kind, it is well-known that more input data usually results in better estimation. In practise, it is not always possible to install a large amount of measuring instruments. Our interest is therefore investing on the minimum number of measurement points needed for the point sources identifications problems for heat equations. We first consider the problem of identifying, from data obtained by a single measurement point b

$$\|u(t,b)-u^{\delta}(t)\|_{L^2(0,T)}\leq\delta, \tag{1.1}$$

the source function of the following heat equation

$$\begin{cases} \partial_t u(x,t)=\Delta u(x,t)+f(x), & x\in\mathbb{R}^d,\ t>0, \\ u(x,0)=0, & x\in\mathbb{R}^d, \end{cases} \tag{1.2}$$

where the source function f is assumed to be a linear combination of dirac-delta function,

$$f(x)=\sum_{\ell=1}^{N}\sigma_\ell\delta(x-a_\ell). \tag{1.3}$$

In [17], we show that (1.2) is equivalent to the one with time dependent source function $h(t)f(x)$ using the Volterra equation of the second kind. Moreover, we analyzed the above problem without noise in two-dimension with the dirac-delta function $\delta(\cdot)$ in (1.3) approximated by some radially symmetric functions in the Schwartz space $\mathscr{S}(\mathbb{R}^2)$ of rapidly decreasing functions. Furthermore, we assumed all strengths are unitary, e.g. $\sigma_\ell=1$ for all ℓ in (1.3). We showed that one measurement point is sufficient to identify the number of sources and three measurement points are sufficient to determine all unknown source positions.

In this paper, the same results are shown to hold under a general setting: besides of the number of sources N and their locations a_ℓ, the strength σ_ℓ ($\ell=1,\cdots,N$) is now considered as considered as unknown. Moreover, the analysis of this paper takes noise into account. The work in this paper can be applied to the formulation in [17] when dirac-delta function $\delta(\cdot)$ is replaced by $\rho\in\mathscr{S}(\mathbb{R}^d)$.

Let b be a measurement point in $\mathbb{R}^d$. Our problem here is to identify the number of sources N, the source strengths σ_ℓ and the locations a_ℓ from the noisy data $u^{\delta}(t)$ at the