Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2017-0187

Vol. **24**, No. 5, pp. 1409-1434
November 2018

Solution of Cauchy Problems by the Multiple Scale Method of Particular Solutions Using Polynomial Basis Functions

Ji Lin[1,*], Yuhui Zhang[1], Thir Dangal[2] and C. S. Chen[3]

[1] *International Center for Simulation Software in Engineering and Sciences, College of Mechanics and Materials, Hohai University, Nanjing 211100, China.*
[2] *Department of Mathematics and Computer Science, Alcorn State University, Lorman, MS 39096, USA.*
[3] *Department of Mathematics, University of Southern Mississippi, Hattiesburg, MS 39406, USA.*

Received 3 September 2017; Accepted (in revised version) 27 October 2017

Abstract. We have recently proposed a new meshless method for solving second order partial differential equations where the polynomial particular solutions are obtained analytically [1]. In this paper, we further extend this new method for the solution of general two- and three-dimensional Cauchy problems. The resulting system of linear equations is ill-conditioned, and therefore, the solution will be regularized by using a multiple scale technique in conjunction with the Tikhonov regularization method, while the L-curve approach is used for the determination of a suitable regularization parameter. Numerical examples including 2D and 3D problems in both smooth and piecewise smooth geometries are given to demonstrate the validity and applicability of the new approach.

AMS subject classifications: 65N21, 65N30

Key words: Method of particular solution, polynomial basis function, multiple scale technique, regularization technique, Cauchy problem.

1 Introduction

Inverse problems arise naturally in many scientific and engineering areas, such as detection of corrosion inside a pipe [2], reconstruction of obstacle in acoustic field [3], nondestructive testing [4], and electrical impedance tomography [5]. In this paper, we will

*Corresponding author. *Email addresses:* `linji861103@126.com`, `linji@hhu.edu.cn` (J. Lin), `zyh940511@outlook.com` (Y. Zhang), `tdangal@alcorn.edu` (T. Dangal), `cschen.math@gmail.com` (C. S. Chen)

focus on the so-called inverse Cauchy problem [6], i.e., the case when the boundary conditions are not completely known, due to technical difficulties associated with data acquisition. We need to reconstruct the solution from over-determined measurements on the accessible part of the boundary. The Cauchy problem is severely ill-posed in the sense that a small perturbation in the given data may have an enormous effect on the numerical solution [7]. Therefore, an accurate and stable solution procedure to the Cauchy problem is of great importance in many engineering applications.

Several numerical methods have been proposed for the solution of such problems [8–10]. Generally speaking, these methods can be divided into two groups: iterative methods and non-iterative methods. For iterative methods, we start by defining an initial guess for the solution and generate a sequence of improved approximate solutions by minimizing certain functionals [11]. These methods could be extremely time-consuming since we have to solve the forward problem at each iterative step. In contrast, non-iterative methods are computationally much less expensive. However, these methods are usually unstable. In recent decades, some classes of alternative methods have been proposed for the simulation of the Cauchy problem, for example, the method of fundamental solutions (MFS) [12–14], the boundary knot method (BKM) [15], the plane waves method [16], the Trefftz method [17,18], and the singular boundary method (SBM) [19,20]. These methods belong to the category of the boundary-type meshless methods which can solve homogeneous problems with a boundary-only discretization. However, they require special methods such as the dual reciprocity method (DRM) [21] to handle inhomogeneous problems. In Ref. [22], Li proposed a non-iterative one-step numerical method based on radial basis functions for solving inverse problems with source terms. In Ref. [24], Karageorghis presented a survey article on applications of the MFS to inverse problems. It was then extended to inverse biharmonic boundary value problems [23], parameter estimation [25], inverse heat conduction problems [26, 27], and inverse reaction-diffusion problems [28].

In this paper, we propose a new non-iterative one-step numerical method to solve the Cauchy problem for linear differential equations with source terms. Our method is based on the recently developed method of particular solutions (MPS) [29] which has been successfully applied to many engineering problems, such as the Navier-Stokes equations [30], the wave propagation problem [31], and the time-fractional diffusion problem [32]. Unlike the traditional approach which applies radial basis functions for approximating particular solutions, we develop a new method which avoids the use of shape parameters in radial basis functions [33]. For this purpose, a polynomial basis function is used instead of radial basis functions to form particular solutions for general partial differential operators with constant coefficients [1]. There are two fundamental issues that arise in the solution of the Cauchy problem using the MPS with polynomials: (1) With the increasing order of polynomials, the collocation matrix becomes highly ill-conditioned which may result in numerical instability and (2): How to deal with the ill-posed Cauchy problem? To alleviate such difficulties, a multiple scale technique [34] in combination with the Tikhonov regularization technique is proposed. For the Tikhonov regulariza-