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A Backward Doubly Stochastic Differential Equation Approach for Nonlinear Filtering Problems

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Abstract. A backward doubly stochastic differential equation (BDSDE) based nonlinear filtering method is considered. The solution of the BDSDE is the unnormalized density function of the conditional expectation of the state variable with respect to the observation filtration, which solves the nonlinear filtering problem through the Kallianpur formula. A first order finite difference algorithm is constructed to solve the BSDES, which results in an accurate numerical method for nonlinear filtering problems. Numerical experiments demonstrate that the BDSDE filter has the potential to significantly outperform some of the well known nonlinear filtering methods such as particle filter and Zakai filter in both numerical accuracy and computational complexity.

AMS subject classifications: 65C30, 65K10

Key words: Nonlinear filtering problems, backward doubly stochastic differential equation, first order algorithm, quasi Monte Carlo sequence

1 Introduction

In this paper, we consider the following nonlinear filtering problem with correlated noises on the probability space $(\Omega,\mathcal{F},P)$

$$dS_t = c(S_t)dt + \tilde{\rho}dW_t + \rho dV_t, \tag{1.1}$$

$$dM_t = h(S_t)dt + dV_t, \tag{1.2}$$

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where the state process $\{S_t\}_{t\geq 0}$ and the observation process $\{M_t\}_{t\geq 0}$ take values in $\mathbb{R}^d$, respectively, and $\mathbb{R}^l$; $c:\mathbb{R}^d\to\mathbb{R}^d$ and $h:\mathbb{R}^d\to\mathbb{R}^l$ are two nonlinear functions; $W_t\in\mathbb{R}^d$ and $V_t\in\mathbb{R}^l$ are two independent Brownian motions; $\tilde{\rho}\in\mathbb{R}^{d\times d}$ and $\rho\in\mathbb{R}^{d\times l}$ are two constant coefficient matrices. The given initial value S_0 is independent from W_t and V_t with probability density p_0. The goal of the stochastic filtering problem is to obtain the best estimate of $\Phi(S_t)$ as the conditional expectation with respect to observation $\{M_s, 0\leq s\leq t\}$ [6,40,53] where Φ is a test function. Assume that $\Phi(S_t)\in L^2(P)$. Then the nonlinear filtering problem can be expressed as finding the stochastic process $\tilde{\Phi}(S_t)$ such that

$$\tilde{\Phi}(S_t)=E[\Phi(S_t)\,|\,\mathcal{M}_t]=\inf\{E[|\Phi(S_t)-K_t|^2];\ K_t\in\mathcal{K}_t)\}, \tag{1.3}$$

where $\mathcal{M}_t=\sigma\{M_s,\ 0<s\leq t\}$ is the σ-algebra generated by the observation process up to t, $\mathcal{K}_t$ is the space of all $\mathcal{M}_t$-measurable and square integrable random variables. When the covariance matrix $\rho=0$, the nonlinear filtering problem (1.1)-(1.2) becomes the standard nonlinear filtering problem.

Examples of filtering problems arise in biology [7,41], mathematical finance [16,25,27, 30], image processing [60], target tracking [36], and many engineering applications. For linear filtering problems, where both c and h are linear functions, the major breakthrough was due to the landmark work of Kalman [39]. For nonlinear filtering problems, the extended Kalman filter (EKF) [14, 38] and the particle filter method (PFM) [3, 9, 19, 22, 31] are two of the widely used methods. In EKF, the state system and the observations are linearized so that the standard Kalman filter can be applied. The PFM, which is essentially a sequential Monte Carlo method, uses a number of independent random variables called particles sampled directly from the state space to represent the prior probability, and updates the prior by including the new observation with the Bayes' theorem to get the posterior. While both EKF and PFM have been successful in solving nonlinear filtering problems, each of them has its drawbacks and limitations. For instance, the EKF performs very poorly when the dynamics are highly nonlinear or the noise intensities are high. For PFM, stability is a challenge for long term simulations or high frequency data (see [24]).

The Zakai filter is another well known nonlinear filtering method. It differs from the EKF and the PFM in that it produces the "exact solution" of the nonlinear filtering problem by providing a stochastic partial differential equation, known as Zakai equation [53], whose solution solves the nonlinear filtering problem through the Kallianpur formula. The Zakai equation is the counter part of the Fokker-Planck equation for an SDE. If solved accurately, the Zakai equation can provide accurate solutions for nonlinear filtering problems [4,11,20,21,23,26,29,32,34,44,53]. However, because of low regularity of the solution of the Zakai equation, the Zakai filter is often prohibitively expensive for practical applications, even when the spatial dimension and the observation dimension of (1.1)-(1.2) are not very high.

In this paper, we propose a novel accurate numerical method for nonlinear filtering problems in which a system of forward backward doubly stochastic differential equations (FBDSDE) is solved. We name this method as BDSDE filter. The Feyman-Kac type