Commun. Comput. Phys. Vol. **23**, No. 4, pp. 899-909
doi: 10.4208/cicp.OA-2016-0239 April 2018

Numerical Evidence of Sinai Diffusion of Random-Mass Dirac Particles

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Received 11 December 2016; Accepted (in revised version) 7 January 2017

Abstract. We present quantum Lattice Boltzmann simulations of the Dirac equation for quantum-relativistic particles with random mass. By choosing zero-average random mass fluctuation, the simulations show evidence of localization and ultra-slow Sinai diffusion, due to the interference of oppositely propagating branches of the quantum wavefunction which result from random sign changes of the mass around a zero-mean. The present results indicate that the quantum lattice Boltzmann scheme may offer a viable tool for the numerical simulation of quantum-relativistic transport phenomena in topological materials.

PACS: 03.65.Pm, 02.70.-c

Key words: Sinai diffusion, Anderson localization, QLB method, Dirac equation, random mass.

1 Introduction

Topological materials, such as insulators and superconductors, have attracted enormous interest in the recent years, as a novel form of quantum matter exhibiting a whole array of fascinating properties [1, 2]. In particular, *disordered* low-dimensional quantum materials present an intriguing interplay between topology and localization, eventually leading to the suppression of Anderson localization in the vicinity of a phase transition between two distinct topological phases [3, 4]. This motivates an intense study of the basic mechanisms which govern the transport properties of low-dimensional quantum materials in the vicinity of a topological phase transitions. Recent work in this direction has highlighted an intriguing tendency to ultra-retarded transport as described by classical *Sinai diffusion* in a random *force* field. Such tendency is, at least at a first glance, quite surprising, for there is no reason a-priori why transport phenomena in low-dimensional

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quantum disordered materials should exhibit any analogy with a classical Langevin particle in a random force field. Such an analogy has been analysed in a recent paper, in the context of quantum transport in Majorana quantum wires [5], based on analytical inspection of the one-dimensional Dirac equation for a free quantum-relativistic particle with a zero-average random mass. In this paper, we analyse a similar problem by means of numerical simulations of the random-mass Dirac equation based on the quantum lattice Boltzmann method [6–8]. Our numerical results confirm strong localization at increasing noise strength, and provide indications of ultra-slow Sinai diffusion as well.

2 Sinai diffusion: from classical to quantum regimes

In 1982 Sinai analyzed the classical motion of a stochastic particle moving in a random *force* field [11],

$$\frac{dz}{dt}=p/m, \tag{2.1}$$

$$\frac{dp}{dt}=-\gamma p+\tilde{F}, \tag{2.2}$$

where $p=mv$ is the particle momentum, γ the friction frequency and $\tilde{F}$ a random force with zero mean and variance D. Analytical treatment of the above Langevin equation, shows that diffusion proceeds at ultra-slow rates, with a variance scaling as:

$$<z^2>=\lambda^2\log^4(t), \tag{2.3}$$

$\lambda\propto 1/D$ being the localization length of the corresponding probability distribution $p(z)$. Such ultra-slow diffusion is basically due to the fact that a random force translates into a relatively regular potential $V(z)$, whose barriers are more effective than random ones in trapping the particle in the corresponding local minima. The first evidences that Sinai diffusion might be relevant to relativistic quantum transport was provided by pioneering work on one-dimensional random mass Dirac fermions and equivalent random Ising chains [14, 15].

Indeed, the analysis of the Dirac equation with a random mass, such that

$$<m(z)>=0, \tag{2.4}$$

$$<m(z)m(z')>=2\lambda^{-1}\delta(z-z') \tag{2.5}$$

shows that the spectrum becomes not only gapless but also quantum critical, i.e. it supports order-disorder transitions driven by purely quantum fluctuations, with no contributions from thermal ones. This can be heuristically understood by nothing that the zero-energy (steady-state) Dirac equation is formally solved by $\psi_{\pm}(z)\propto e^{\mp\int_{-\infty}^{z} m(z')dz'}$, so that, away from null points, where $m(z^*)=0$, the critical solutions decay like $e^{-\sqrt{|z-z^*|/\lambda}}$. For small positive-energy solutions $E>0$, the behaviour remains similar, and it can be