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A High-Order Method for Weakly Compressible Flows

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Abstract. In this work, we introduce an IMEX discontinuous Galerkin solver for the weakly compressible isentropic Euler equations. The splitting needed for the IMEX temporal integration is based on the recently introduced *reference solution* splitting [32, 52], which makes use of the *incompressible* solution. We show that the overall method is *asymptotic preserving*. Numerical results show the performance of the algorithm which is stable under a convective CFL condition and does not show any order degradation.

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1 Introduction

In this work, we consider the (weakly-)compressible isentropic Euler equations [2, 59] in dimensionless form,

$$\begin{aligned} &\rho_t + \nabla\cdot(\rho \boldsymbol{u}) = 0, \\ &(\rho \boldsymbol{u})_t + \nabla\cdot(\rho \boldsymbol{u}\otimes \boldsymbol{u}) + \frac{1}{\varepsilon^2}\nabla p = 0. \end{aligned} \tag{1.1}$$

The wave speeds in normal direction $\boldsymbol{n}$ of this (assumed two-dimensional) problem are

$$\lambda_1 = \boldsymbol{u}\cdot\boldsymbol{n} \qquad \text{and} \qquad \lambda_{2,3} = \boldsymbol{u}\cdot\boldsymbol{n} \pm \frac{c}{\varepsilon}, \tag{1.2}$$

which means that there is a convective and two acoustic waves. In what follows, we assume that the reference Mach number ε is small, i.e., $\varepsilon \ll 1$, and all the other quantities are

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$\mathcal{O}(1)$, which physically means that the solution is a small disturbance of the incompressible solution. Indeed, it can be shown that under suitable requirements on initial and boundary data ("well-preparedness"), there is convergence of density and momentum $(\rho,\rho\boldsymbol{u})$ towards its incompressible counterpart as $\varepsilon\to 0$, see [35, 51, 61] and the references therein. Furthermore, it is obvious that this problem constitutes a *singularly perturbed equation* in ε, as the equations change type in the limit.

Due to the change of type as $\varepsilon\to 0$, the equations get extremely stiff and therefore it is highly non-trivial to design efficient and stable algorithms. Explicit-in-time solving techniques have the drawback that they lead to a CFL condition in which the time step size Δt must be proportional to $\varepsilon\Delta x$, where Δx is a measure for the spatial grid size. If it is not the goal to accurately resolve all the features, but only to resolve the convective part of the flow, this condition is extremely restrictive, and a so called *convective CFL condition*

$$\Delta t \lesssim \frac{\Delta x}{\|\boldsymbol{u}\|} \tag{1.3}$$

is preferable. Fully implicit-in-time methods, on the other hand, which are stable under such a CFL condition, tend to add too much spurious diffusion [37].

In the past few years, so called IMEX (implicit-explicit) splitting schemes got more and more popular for solving compressible flow problems, especially for low Mach numbers, see e.g. [9, 10, 19, 20, 24, 26, 36, 39, 41, 46, 60] and the references therein. Optimally, such a scheme should be designed in a way that slow waves are handled with an explicit (thus efficient) and fast waves are handled with an implicit (thus unconditionally stable) method. Of course such a strict splitting of waves is only possible in the linear one-dimensional case [53], and therefore, a suitable splitting for the nonlinear multidimensional case has to be defined very carefully.

Over the past few years, many famous splittings for the Euler equations at low Mach number have been designed, beginning by the ground-breaking work of Klein [36]. For a non-exhaustive list, we refer to [9, 20, 26] and the references therein. However, many of those splittings have their shortcomings. It has been reported [63] that Klein's splitting seems to be unstable in some instances. (Which does not include Klein's original algorithm as it is based on a semi discrete decoupling of the pressure.) Furthermore, all of the mentioned splittings need a physical intuition and are not directly extendable to other singularly perturbed differential equations.

To partly overcome these shortcomings, we have over the past few years developed a new type of splitting that is based on the $\varepsilon=0$ ("incompressible") solution of the problem. The splitting, termed RS-IMEX (see Section 3), is generic in the sense that it can in principle be applied to any type of singularly perturbed equation, including singularly perturbed ODEs [52] and the isentropic Euler equations [32]. Related ideas have already been published earlier, for the shallow water equations in [9, 23] and for kinetic equations in [22], a stability analysis of the splitting has been done in [63] and [62].

In [52], we have applied the splitting idea to singularly perturbed ordinary differential equations with high-order IMEX discretizations, namely IMEX linear multistep