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The Discrete Orthogonal Polynomial Least Squares Method for Approximation and Solving Partial Differential Equations

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Abstract. We investigate numerical approximations based on polynomials that are orthogonal with respect to a weighted discrete inner product and develop an algorithm for solving time dependent differential equations. We focus on the family of super Gaussian weight functions and derive a criterion for the choice of parameters that provides good accuracy and stability for the time evolution of partial differential equations. Our results show that this approach circumvents the problems related to the Runge phenomenon on equally spaced nodes and provides high accuracy in space. For time stability, small corrections near the ends of the interval are computed using local polynomial interpolation. Several numerical experiments illustrate the performance of the method.

AMS subject classifications: 41A10, 65D15, 65M70

Key words: Discrete least-squares, orthogonal polynomials, spectral methods, high order numerical methods, uniform grid.

1 Introduction

This paper investigates a high order numerical method for approximating smooth functions on a uniform grid and solving partial differential equations on a hybrid grid in $[-1,1]$. The method uses the discrete orthogonal polynomial least squares (DOP-LS) approximation based on the *super Gaussian weight function*, which is both smoothly connected to zero at ± 1 and equals one in nearly the entire domain. As a result, the method has fast decaying expansion coefficients and also successfully suppresses Runge oscillations that pollute the boundary regions. Such desirable weight function features were

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first exploited in [17] in the context of spectral reprojection from (pseudo-)spectral Fourier data, and later in [15] as a least squares approximation technique for piecewise smooth functions given equally or arbitrarily spaced points. In [17], the Fourier coefficients were reprojected onto the Freud polynomial basis (what we will refer to as a super Gaussian polynomial basis) to eliminate the Gibbs phenomenon. The concept of reprojection from the Fourier basis onto another basis to remove the Gibbs phenomenon has been discussed at length in the context of Gegenbauer reconstruction, see [20,21] and references therein. The Gibbs phenomenon is removed due to the reprojection polynomial weight function being smoothly connected to zero near the boundaries, which prevents the Gibbs oscillations in the Fourier approximation from entering the reprojection and allows rapid decay of the reprojection expansion coefficients. However, the Gegenbauer polynomials are not entirely satisfactory as a reprojection basis due to their high propensity to round-off error. Furthermore, for large orders, the Gegenbauer partial sum expansion behaves like a power series, yielding what was coined the generalized Runge phenomenon in [2]. In contrast, as mentioned above, the super Gaussian weight functions are designed to be one in nearly the entire domain of approximation, so that the growth of the corresponding polynomials is better controlled. The approximation also utilizes more information from the underlying function. In [15] it was noted that the values given on equidistant grid points need not first be converted to pseudo-spectral Fourier coefficients in order to recover a highly accurate approximation. The resulting super Gaussian *discrete orthogonal polynomial least squares* (DOP-LS) method was shown to be robust and efficient for the approximation of smooth functions.

This investigation further analyzes the super Gaussian DOP-LS approximation of smooth functions in $[-1,1]$ when the function is known at uniform grid points. We extend the analysis from [15] to characterize the optimal parameters needed for convergence in $[-1,1]$, as well as in smaller intervals $[-\delta,\delta]$, $0<\delta<1$. This information is then used to develop a new hybrid multi-domain method for the approximation of smooth functions. The technique consists of "patching" the super Gaussian approximation in $[-\delta,\delta]$ with Chebyshev (interpolatory) approximations in the two smaller boundary regions $[-1,-\delta]$ and $[\delta,1]$ on Gauss Lobatto grids. The combined method enables high order approximation of smooth functions with less point clustering than the typical orthogonal polynomial approximation methods.

In the second part of this paper we incorporate the hybrid multi-domain approximation into a numerical method that computes partial differential equations with smooth solutions. Fourier pseudo-spectral methods are well suited for solving periodic smooth problems on discrete data. Orthogonal polynomials, such as Chebyshev or Legendre polynomials, are used as basis polynomials for spectral methods solving smooth non-periodic problems. In this case, the grid points must be distributed so that the quadrature used (typically Gauss or Gauss-Lobatto) to calculate the expansion coefficients yields high enough accuracy. Such distributions are always clustered at the ends of the intervals. This is a traditional bottleneck when solving partial differential equations with spectral methods, since explicit time stepping methods require very small time steps on the order