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High Order Numerical Schemes for Second-Order FBSDEs with Applications to Stochastic Optimal Control

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Abstract. This is one of our series papers on multistep schemes for solving forward backward stochastic differential equations (FBSDEs) and related problems. Here we extend (with non-trivial updates) our multistep schemes in [W. Zhao, Y. Fu and T. Zhou, SIAM J. Sci. Comput., 36 (2014), pp. A1731-A1751] to solve the second-order FBSDEs (2FBSDEs). The key feature of the multistep schemes is that the Euler method is used to discretize the forward SDE, which dramatically reduces the entire computational complexity. Moreover, it is shown that the usual quantities of interest (e.g., the solution tuple (Y_t,Z_t,A_t,Γ_t) of the 2FBSDEs) are still of high order accuracy. Several numerical examples are given to show the effectiveness of the proposed numerical schemes. Applications of our numerical schemes to stochastic optimal control problems are also presented.

AMS subject classifications: 60H35, 65H20, 65H30

Key words: Multistep schemes, second-order forward backward stochastic differential equations, stochastic optimal control.

1 Introduction

This work is concerned with numerical methods for the following coupled second-order forward backward stochastic differential equations (2FBSDEs) which is defined on the filtered probability space $(\Omega,\mathscr{F},\mathbb{F},P)$:

$$\begin{cases} X_t = x+\int_0^t b(s,\Theta_s)\mathrm{d}s+\int_0^t \sigma(s,\Theta_s)\mathrm{d}W_s, \\ Y_t = g(X_T)+\int_t^T f(s,\Theta_s)\mathrm{d}s-\int_t^T Z_s\mathrm{d}W_s, \qquad t\in[0,T], \\ Z_t = Z_0+\int_0^t A_s\mathrm{d}s+\int_0^t \Gamma_s\mathrm{d}W_s, \end{cases} \tag{1.1}$$

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where $\Theta_t=(X_t,Y_t,Z_t,\Gamma_t)\in\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}^d$ is the unknown, $(\Omega,\mathscr{F},P)$ is the given probability space, $T>0$ is the deterministic terminal time, $\{W_t\}_{t\in[0,T]}$ is a d-dimensional Brownian motion defined on $(\Omega,\mathscr{F},P)$ with the natural filtration $\mathbb{F}=\{\mathscr{F}_t\}_{0\le t\le T}$ and all P-null sets in $\mathscr{F}_0$, $x\in\mathscr{F}_0$ is the initial condition of the forward SDE, $\mathcal{S}^d$ is the set of all $d\times d$ real-valued symmetric matrices, and

$$b\colon\ [0,T]\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}^d\to\mathbb{R}^m,\qquad \sigma\colon\ [0,T]\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}^d\to\mathbb{R}^{m\times d}$$

are referred to the drift and diffusion coefficients of the forward SDE, respectively. While

$$f\colon\ [0,T]\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}^d\to\mathbb{R}\quad\text{and}\quad g\colon\ \mathbb{R}^m\to\mathbb{R}$$

are referred to the generator and the terminal condition of the backward SDE, respectively. The three stochastic integrals with respect to the Brownian motion $\{W_t\}_{t\in[0,T]}$ are of the Itô type. A 5-tuple $(X_t,Y_t,Z_t,A_t,\Gamma_t)$ is called an L^2-adapted solution of the 2FBSDEs (1.1) if it is $\mathscr{F}_t$-adapted, square integrable, and satisfies (1.1). Moreover, the 2FBSDEs (1.1) is called *decoupled* if b and σ are independent of Y_t, Z_t A_t and Γ_t.

The 2FBSDEs (1.1), as an extension of the BSDEs in the linear [2] or nonlinear cases [17], was first introduced by Cheridito, Soner, Touzi and Victoir in [5], with a slightly different (yet equivalent) formula, and was further investigated by Soner, Touzi and Zhang in [19]. The main motivation there is to give a precise connection between the 2FBSDEs and fully non-linear PDEs, in particular the Hamilton-Jacobi-Bellman equations and the Bellman-Isaacs equations which are widely used in stochastic control and in stochastic differential games. Such a connection leads to interesting stochastic representation results for fully nonlinear PDEs, generalizing the original (nonlinear) Feynman-Kac representations of linear and semi-linear parabolic PDEs, see e.g. [14,18] and references therein.

From the numerical methods point of view, one can adopt these connections between PDEs and FBSDEs to design the so called probabilistic numerical methods for PDEs, by solving the equivalent FBSDEs (or 2FBSDEs). While there are a lot of work dealing with numerical schemes for BSDEs [1,3,4,21,22,24], however, there are only a few work on numerical methods for FBSDEs [6,8,9,12,13,15,20,23,25,26] and fully non-linear PDEs [7,10]. Some of the above works are designed with high order accuracy that can however only be used to deal with low dimensional FBSDEs, while some of them are low order numerical methods that are suitable for solving high dimensional problems. In particular, we mention the work [10], where a numerical example for a 12-dimensional coupled FBSDE is reported, and it is shown by numerical test that the numerical method converges with order 1. Also, in [9], multistep schemes were proposed to solve multi-dimensional FBSDEs by using the sparse grid interpolation technique, and several multi-dimensional examples with dimension up to 6 are presented, and high order convergence rates up to 3 were obtained.

To the best of our knowledge, there is no related study for high order numerical methods for 2FBSDEs. The main purpose of this work is to extend our multistep schemes in [23] (which is original designed for solving FBSDEs) to the use of solving 2FBSDEs.