Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2016-0029

Vol. **21**, No. 3, pp. 718-747
March 2017

Projected Finite Elements for Systems of Reaction-Diffusion Equations on Closed Evolving Spheroidal Surfaces

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Received 18 February 2016; Accepted (in revised version) 20 July 2016

Abstract. The focus of this article is to present the projected finite element method for solving systems of reaction-diffusion equations on evolving closed spheroidal surfaces with applications to pattern formation. The advantages of the projected finite element method are that it is easy to implement and that it provides a conforming finite element discretization which is "logically" rectangular. Furthermore, the surface is not approximated but described exactly through the projection. The surface evolution law is incorporated into the projection operator resulting in a time-dependent operator. The time-dependent projection operator is composed of the radial projection with a Lipschitz continuous mapping. The projection operator is used to generate the surface mesh whose connectivity remains constant during the evolution of the surface. To illustrate the methodology several numerical experiments are exhibited for different surface evolution laws such as uniform isotropic (linear, logistic and exponential), anisotropic, and concentration-driven. This numerical methodology allows us to study new reaction-kinetics that only give rise to patterning in the presence of surface evolution such as the *activator-activator* and *short-range inhibition; long-range activation*.

AMS subject classifications: 65M12, 65M60, 35K57, 35K58, 9208

Key words: Time-dependent projection operators, projected finite elements, non-autonomous partial differential equations, reaction-diffusion systems, evolving surfaces, Turing *diffusively-driven* instability, pattern formation.

1 Introduction

Currently, there is a surge in modelling parabolic partial differential equations on evolving arbitrary complex surfaces [1, 8, 9, 12–14, 24, 25] partly due to the advances in tech-

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niques for obtaining 3-dimensional datasets in experimental sciences [19]. Typical examples include molecular analysis of the fungus rice blast in plant biology [5], biochemical and biomechanical analysis of cell migration in cell motility [2, 35] and pattern formation in developmental biology [20, 23, 30]. In most cases, molecular species resident on the cell-surface are observed to react and diffuse and in doing so, induce surface evolution. Mathematical modelling of such processes results in highly nonlinear systems of reaction-diffusions equations posed on time-dependent closed (and sometimes open) surfaces or manifolds. The theory of reaction-diffusion is well-studied on stationary planar domains but not so much is known of the models on evolving planar domains and surfaces. Due to the nature of the nonlinearities, closed form solutions are not readily available. Furthermore, including surface evolution adds extra complexities to the model system in that non-autonomous systems of parabolic partial differential equations are obtained which render redundant standard theoretical analytical techniques such as the linear stability theory for analysing the model dynamics close to the bifurcation points [17, 27]. Hence it is critical to develop new numerical methodologies and techniques for solving, robustly and efficiently, systems of non-autonomous nonlinear parabolic partial differential equations on complex evolving surfaces.

To-date, there has been an increase in the development of numerical methods for approximating solutions of partial differential equations posed on evolving surfaces. Examples include (but are not limited to) the method of lines [4], evolving surface finite element methods on triangulated surfaces [1, 8, 9, 12, 13], implicit finite element methods using level set descriptions of the surfaces [11, 12, 33, 37], diffuse interface methods of which phase-fields are an example [3, 6, 14], particle methods using level set descriptions of the surface [7, 16, 18, 22] and closest-point methods [24, 25]. In all these methods the continuous surface is approximated by a discrete surface thereby committing a *geometrical error*. A key issue is how the surface description is encoded into the numerical method. For the evolving surface finite elements, the surface is approximated by a triangulated surface. The geometrical description of the surface is encoded through the knowledge of the vertices of the triangulation. A geometrical error is committed in carrying out the surface triangulation [13]. On the other hand, numerical methods based on implicit surfaces require the knowledge of the level set function that defines the surface geometry [13]. A key difference between these methods and the projected finite element method (PFEM) is that the latter does not commit geometric errors since the surface is not approximated but described exactly through the projection. The surface evolution is embedded into the projection operator.

The PFEM proposed in this article is inspired by the radially projected finite element method which was used to compute approximate numerical solutions for partial differential equations on stationary spheroidal surfaces such as spheres, ellipsoids, and tori [31, 38, 39]. The PFEM gives a geometrically exact discretization of spheroidal surfaces (the geometry is not approximated, but represented exactly) and is attractive for numerical simulations since the resulting finite element discretization is conforming and is "logically rectangular." The PFEM is easy to implement and incorporate into existing