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Galerkin Method for Wave Equations with Uncertain Coefficients

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Abstract. Polynomial chaos methods (and generalized polynomial chaos methods) have been extensively applied to analyze PDEs that contain uncertainties. However this approach is rarely applied to hyperbolic systems. In this paper we analyze the properties of the resulting deterministic system of equations obtained by stochastic Galerkin projection. We consider a simple model of a scalar wave equation with random wave speed. We show that when uncertainty causes the change of characteristic directions, the resulting deterministic system of equations is a symmetric hyperbolic system with both positive and negative eigenvalues. A consistent method of imposing the boundary conditions is proposed and its convergence is established. Numerical examples are presented to support the analysis.

AMS subject classifications: 65C20, 65C30

Key words: Generalized polynomial chaos, stochastic PDE, Galerkin method, hyperbolic equation, uncertainty quantification.

1 Introduction

In recent years there is a growing interests in studying efficient numerical methods for solving differential equations with random inputs. The Polynomial Chaos (PC) based methods have received intensive attention. The original PC method was developed by R. Ghanem, cf. [3], and was inspired by the Wiener chaos expansion which uses Hermite polynomials of Gaussian random variables to represent random processes [5]. Later the approach was extended to generalized Polynomial Chaos (gPC) where general orthogonal polynomials are adopted for improved representations of more general random processes [7]. With PC/gPC serving as a complete basis to represent random processes, a

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stochastic Galerkin projection can be used to transform the (stochastic) governing equations to a set of deterministic equations that can be readily discretized via standard numerical techniques. Although such a Galerkin approach is effective in many problems, see, e.g., [2, 6, 8], its application to hyperbolic problems has been limited as of now. We believe that the primary reason is that the properties of the system of equations resulting from a Galerkin projection is not fully understood. (When the uncertainty does not change the direction of the characteristics, the Galerkin system can be shown to be hyperbolic and solved in a straightforward manner [1]).

We discuss in this paper the application of the gPC Galerkin method to the simulations of hyperbolic systems that contain uncertainties. In general these uncertainties may enter through initial conditions, boundary conditions or through uncertainties in the coefficients of the problem. Here we deal with the case that the coefficients are functions of random variables. In particular we use a scalar wave equation as a model and study the situation in which the inflow-outflow conditions change as a function of a random variable. The problem is whether it is possible to impose boundary conditions on the deterministic system, consistent with the boundary conditions of the original equation.

We show, in this paper, that the deterministic system is a symmetric hyperbolic system with positive as well as negative eigenvalues. A consistent and stable method of imposing the boundary conditions is outlined. The boundary conditions are not satisfied exactly at the boundaries but rather to the order of the scheme. Convergence of the scheme is established.

The paper is organized as follows. In Section 2 we present the model problem of a scalar hyperbolic equation where the wave speed is a random variable. A consistent set of boundary conditions are presented for the deterministic system resulted from a gPC Galerkin procedure, and we prove convergence of the scheme. In Section 3 we present numerical results to support the theory.

2 Model problem: Scalar wave equation with uncertainty

A simple scalar equation that illustrates the difficulties in applying the (generalized) Polynomial Chaos to hyperbolic equations is:

$$\frac{\partial u(x,t,y)}{\partial t} = c(y)\frac{\partial u(x,t,y)}{\partial x}, \quad x \in (-1,1), \quad t > 0, \tag{2.1}$$

where $c(y)$ is a random transport velocity of a random variable $y \in \Omega$ in a properly defined complete random space with event space Ω and probability distribution function $\rho(y)$. With this the expectation of a given function is

$$\mathbb{E}[f(y)] = \int f(y)\rho(y)dy.$$

At this stage we would like to mention that we can consider (2.1) as a system where c is a symmetric matrix and obtain similar results. For simplicity we stay with the example