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A Finite Volume Scheme for Three-Dimensional Diffusion Equations

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Abstract. The extension of diamond scheme for diffusion equation to three dimensions is presented. The discrete normal flux is constructed by a linear combination of the directional flux along the line connecting cell-centers and the tangent flux along the cell-faces. In addition, it treats material discontinuities by a new iterative method. The stability and first-order convergence of the method is proved on distorted meshes. The numerical results illustrate that the method appears to be approximate second-order accuracy for solution.

AMS subject classifications: 65M08, 65M12, 65M55

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1 Introduction

Accurate and efficient discretization methods for the diffusion equation on distorted meshes are very important for the numerical simulations of Lagrangian hydrodynamics and magnetohydrodynamics. The finite volume method is widely used in solving these practical problems.

Many finite volume schemes for solving diffusion equations on non-rectangular meshes have been proposed. By using integral interpolation method, a nine point scheme for two-dimensional diffusion equation on arbitrary quadrilateral meshes is constructed in [1]. This scheme has only cell-centered unknowns after cell-vertex unknowns are eliminated by taking them as the arithmetic average of the neighboring cell-centered

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unknowns. Numerical experiments show that it loses remarkably accuracy on moderately and highly skewed meshes. In [2] a continuous edge flux scheme for the stationary diffusion equation with smooth coefficient is proposed. The scheme in [3] needs both cell-centered unknowns and vertex unknowns on staggered meshes, and then a generalization of this scheme is proposed in [4] that can be applied to diffusion problems with discontinuous tensor coefficients. Later in [5], the convergence of the scheme of [3] is proved and the construction of the scheme is further extended to the diffusion problem with discontinuous coefficients on staggered meshes, moreover the discontinuity on cell edges is treated rigorously. The MDHW scheme in [6] rigorously treats material discontinuities and yields second-order accuracy regardless of the smoothness of the mesh. This scheme has face-centered unknowns in addition to cell-centered unknowns and the resulted matrix is asymmetric.

The support operators method (SOM) in [7–9] gives second-order accuracy on both smooth and non-smooth meshes either with or without material discontinuities, and SOM generally leads to a symmetric positive definite matrix. However, SOM has both cell-centered and face-centered unknowns or has a dense diffusion matrix, and there has no explicit discrete expression for the normal flux on a cell edge. The multipoint flux approximation (MPFA) method in [10–12] leads to a nonsymmetric matrix. It has only cell centered unknowns and gives an explicit expression for the face-centered flux. The flux is approximated by a multipoint flux expression based on transmissibility coefficients. These coefficients are computed by locally solving a small linear system.

Some successive works are devoted to the development of the schemes mentioned above. A similar finite volume scheme is proposed in [13] by using variation principle. To improve the accuracy, a method of eliminating vertex unknowns in the nine point scheme on distorted quadrilateral meshes is presented in [14], where the vertex unknowns are expressed as the interpolation of neighboring cell-centered unknowns based on certain rigorous derivation. The resulting scheme has only cell-centered unknowns and obtains second-order accuracy, however it leads to a nonsymmetric matrix in general. A different method of calculating the vertex unknowns of nine point scheme is proposed in [15], in which the vertex unknowns are solved independently on a new Voronoï mesh. This method is suitable for solving diffusion problems with discontinuous coefficients on highly distorted meshes and it is proved that this scheme has first-order convergence on distorted meshes.

Some finite volume schemes with full pressure support are presented in [16] and [17], where a method of M-matrix analysis is used to identify bounding limits for the scheme to posses a local discrete maximum principle and the conditions for the scheme to be positive definite are also derived. Based on SOM scheme, a mimetic finite difference method is proposed in [18,19]. This method has been shown to be second-order accurate on nonsmooth quadrilateral meshes with hanging nodes in both Cartesian and $r-z$ geometries.

For solving three-dimensional diffusion equation on distorted meshes, some cell-centered discrete schemes are discussed in [20]- [29]. Based on [1], a discrete scheme on the irregular hexahedral meshes is constructed in [20] by using integral interpolation