

Fast Solution for Solving the Modified Helmholtz Equation with the Method of Fundamental Solutions

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Abstract. The method of fundamental solutions (MFS) is known as an effective boundary meshless method. However, the formulation of the MFS results in a dense and extremely ill-conditioned matrix. In this paper we investigate the MFS for solving large-scale problems for the nonhomogeneous modified Helmholtz equation. The key idea is to exploit the exponential decay of the fundamental solution of the modified Helmholtz equation, and consider a sparse or diagonal matrix instead of the original dense matrix. Hence, the homogeneous solution can be obtained efficiently and accurately. A standard two-step solution process which consists of evaluating the particular solution and the homogeneous solution is applied. Polyharmonic spline radial basis functions are employed to evaluate the particular solution. Five numerical examples in irregular domains and a large number of boundary collocation points are presented to show the simplicity and effectiveness of our approach for solving large-scale problems.

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1 Introduction

One of the major advantages of boundary element methods (BEMs) over finite element (FEM), finite difference (FDM) and finite volume methods (FVM) is their ability to transform the domain integral into the boundary and thus avoid domain discretization which

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is often the most tedious and expensive part of the solution process. However, for inhomogeneous problems, domain integration is required in the formulation of BEMs which takes away their main advantage. During the past two decades, much effort in the BEM literature has been devoted to this issue with great success. The most notable schemes in this direction are the dual reciprocity method (DRM) [1] and the multiple reciprocity method (MRM) [2]. The DRM is in fact a process of evaluating the particular solution without direct numerical integration and is equivalent to the method of particular solutions (MPS). We will use the MPS due to its close connection with the method of fundamental solutions (MFS) which is the focus of this paper. Despite the advantage of mesh reduction by one dimension, the resultant matrix in the BEM formulation is dense in contrast to the sparse matrices obtained with traditional methods such as the FEM, FDM and FVM. Hence, the second major challenge for BEMs is how to overcome the necessity to solve the resulting dense systems.

Since the early 1990s, the method of fundamental solutions has re-emerged as an effective meshless method. Instead of boundary discretization as in the classical BEM, only the boundary collocation points are used in the solution process. The MFS is attributed to Kupradze in 1964 [3] and is classified as an indirect boundary method or regular BEM in the engineering literature. In the MFS, the singularity is avoided by the use of a fictitious boundary outside the domain of interest. As a result, the MFS has the following advantages over the classical BEM: (i) It requires no boundary discretization. (ii) No boundary integration is required. (iii) It converges exponentially for smooth boundary shapes and boundary data. (iv) It is attractive for solving high dimensional problems. (iv) Its implementation and coding are easy. Despite all these attractive features, the MFS was not considered as a main-stream numerical method due to its limitation in solving only homogeneous problems and the uncertainty in choosing the fictitious boundary. An important reason for which the MFS has gradually received attention from the science and engineering community is that, due to the effort of Golberg and Chen [4], it has been successfully extended to solving nonhomogeneous problems and various types of time-dependent problems by being used in conjunction the MPS. With the combined features of the MFS and the MPS, a truly meshless numerical scheme (MFS-MPS) for solving partial differential equations can be obtained. In the MFS-MPS, two dense matrix systems, one for finding the particular solution using the MPS and the other for obtaining the homogeneous solution using the MFS, need to be solved. The development of the compactly supported radial basis functions (CS-RBFs) [5] has made it possible for the matrix in the MPS to be sparse [6]. However, no progress has been reported in the effort to formulate a sparse matrix in the context of the MFS. It is desirable that MFS-MPS has the combined features of 'sparsity' and 'meshlessness'.

It is the purpose of this paper to investigate, apparently for the first time, how a sparse formulation of the MFS for the modified Helmholtz equation, which has wide applications in time-dependent PDEs [7], can be achieved.