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Exponential Time Differencing Gauge Method for Incompressible Viscous Flows

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Abstract. In this paper, we study an exponential time differencing method for solving the gauge system of incompressible viscous flows governed by Stokes or Navier-Stokes equations. The momentum equation is decoupled from the kinematic equation at a discrete level and is then solved by exponential time stepping multistep schemes in our approach. We analyze the stability of the proposed method and rigorously prove that the first order exponential time differencing scheme is unconditionally stable for the Stokes problem. We also present a compact representation of the algorithm for problems on rectangular domains, which makes FFT-based solvers available for the resulting fully discretized system. Various numerical experiments in two and three dimensional spaces are carried out to demonstrate the accuracy and stability of the proposed method.

AMS subject classifications: 65M06, 65M22, 65Y20, 76D05, 76D07

Key words: Incompressible flows, Stokes equations, Navier-Stokes equations, gauge method, exponential time differencing.

1 Introduction

As a fundamental model of incompressible viscous flows, the time-dependent parabolic system of the velocity field $\mathbf{u}(t,\mathbf{x})=(u_1,\cdots,u_d)$ and the pressure $p(t,\mathbf{x})$,

$$\begin{cases} \mathbf{u}_t-\nu\Delta\mathbf{u}+\mathbf{F}(\mathbf{u})+\nabla p=\mathbf{f}, & \text{in } [0,T]\times\Omega, \\ \nabla\cdot\mathbf{u}=0, & \text{in } [0,T]\times\Omega \end{cases} \tag{1.1}$$

has wide applications in engineering and scientific problems. In the mathematical model, $\Omega\in\mathbb{R}^d$ is the domain, $\mathbf{f}=(f_1,\cdots,f_d)$ represents the body force, $\mathbf{F}=(F_1,\cdots,F_d)$ represents

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the nonlinear convection, and $\nu>0$ denotes the kinematic viscosity of the fluids. When $\mathbf{F}(\mathbf{u})$ is zero, the system is Stokes; when $\mathbf{F}(\mathbf{u})=(\mathbf{u}\cdot\nabla)\mathbf{u}$, the system is Navier-Stokes. Many numerical methods have been developed for solving the system (1.1) in order to simulate, predict and/or control the flows (see [10, 11, 14, 15, 25, 33] and references cited therein).

In the development of efficient time integration methods for the fluid system, special attentions have been drawn to deal with the incompressibility constraint. One of the most popular methods is the projection method (or the so-called fractional step methods), which was first developed in the late 1960s by Chorin and Temam independently [4, 32]. The basic idea is to decouple the velocity and pressure in a discrete setting so that one only needs to solve a sequence of elliptic equations. Thus, it would greatly reduce the computational complexity compared to the original fully coupled system. The existing projection methods are usually classified into three categories [11]: the pressure-correction methods, the velocity-correction methods, and the consistent splitting methods. Among them, the popular pressure-correction methods ignore or treat explicitly the pressure term in the first sub-step (i.e., treat viscous effect only) and then correct it in the second sub-step (i.e., treat incompressibility); the velocity-correction methods switch the roles of velocity and pressure terms as those in the pressure-correction method. In this approach, the viscous effect is ignored or treated explicitly in the first sub-step and then corrected in the second one; the consistent splitting methods first compute the velocity by treating the pressure explicitly, then update the pressure by using the weak form of a Poisson equation for the pressure. Although these approaches have been widely used, it is still difficult to develop high-order (in time) schemes for both the velocity and pressure. One of the main reasons is that the boundary condition for the pressure equation in projection methods is artificial, which limits the flexibility and accuracy of the projection methods, especially for the pressure approximation.

Another splitting approach is the gauge method [6–8, 29–31]. The method is based on the Hodge decomposition (or Helmholtz-Hodge decomposition), which states that a sufficient smooth, rapidly decaying vector field $\mathbf{m}=(m_1,\cdots,m_d)$ can be decomposed into the sum of a divergence-free term $\mathbf{u}$ (a solenoidal part) and the gradient of a scalar potential ϕ (an irrotational part), i.e.,

$$\mathbf{m}(t,\mathbf{x})=\mathbf{u}(t,\mathbf{x})+\nabla\phi(t,\mathbf{x}), \tag{1.2}$$

where $\nabla\cdot\mathbf{u}=0$ and these two components are orthogonal. The gauge system is reformulated from (1.1), in which the velocity field $\mathbf{u}$ and the pressure p are replaced by the auxiliary filed $\mathbf{m}$ and the gauge variable ϕ. Based on the Hodge decomposition (1.2) and the boundary conditions of velocity $\mathbf{u}$, certain simple but consistent boundary conditions can be assigned for both $\mathbf{m}$ and ϕ. The resulting system consists of a second-order parabolic problem of $\mathbf{m}$ and a Poisson problem of ϕ that are weakly coupled through the boundary conditions. In order to fully decouple the auxiliary field from the gauge variable during simulations, an explicit extrapolation was used to generate an approximation of the boundary values of $\mathbf{m}$ at a current time step by using its approximations from previous