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Long Time Behaviour of an Exponential Integrator for a Vlasov-Poisson System with Strong Magnetic Field

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Abstract. With the aim of solving in a four dimensional phase space a multi-scale Vlasov-Poisson system, we propose in a Particle-In-Cell framework a robust time-stepping method that works uniformly when the small parameter vanishes. As an exponential integrator, the scheme is able to use large time steps with respect to the typical size of the solution's fast oscillations. In addition, we show numerically that the method has accurate long time behaviour and that it is asymptotic preserving with respect to the limiting Guiding Center system.

AMS subject classifications: 65L04, 65M99, 35L03, 70B05

Key words: Vlasov-Poisson system, Guiding-Center, Particle-In-Cell method, highly oscillatory ODEs, long-time simulation.

1 Introduction

In this paper we introduce a numerical scheme in order to simulate efficiently in time when the parameter ε vanishes the following four dimensional Vlasov equation

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$$\partial_t f^\varepsilon + \mathbf{v}\cdot\nabla_{\mathbf{x}} f^\varepsilon + \left(\Xi^\varepsilon + \frac{1}{\varepsilon}\mathbf{v}^\perp\right)\cdot\nabla_{\mathbf{v}} f^\varepsilon = 0, \tag{1.1}$$

$$f^\varepsilon(\mathbf{x},\mathbf{v},t=0) = f_0(\mathbf{x},\mathbf{v}), \tag{1.2}$$

where $\mathbf{x}=(x_1,x_2)$ stands for the position variable, $\mathbf{v}=(v_1,v_2)$ for the velocity variable, $\mathbf{v}^\perp$ for $(v_2,-v_1)$, $f^\varepsilon \equiv f^\varepsilon(\mathbf{x},\mathbf{v},t)$ is the distribution function, f_0 is given, and $\Xi^\varepsilon \equiv \Xi^\varepsilon(\mathbf{x},t)$ corresponds to the electric field. Weak-$*$ and two-scale limits when ε goes to zero of this equation can be rigorously obtained following the methods introduced in [2] and [13]. We notice that Eqs. (1.1)-(1.2) can be obtained from the six dimensional drift-kinetic regime by taking a constant magnetic field in the x_3-direction and an electric field evolving in the orthogonal plane to the magnetic field.

The main application will be the case when the electric field Ξ^ε is obtained by solving the Poisson equation. In this case we will rather denote by $\mathbf{E}^\varepsilon$ the electric field and thus, we will have to solve the following nonlinear system of equations:

$$\partial_t f^\varepsilon + \mathbf{v}\cdot\nabla_{\mathbf{x}} f^\varepsilon + \left(\mathbf{E}^\varepsilon + \frac{1}{\varepsilon}\mathbf{v}^\perp\right)\cdot\nabla_{\mathbf{v}} f^\varepsilon = 0, \tag{1.3}$$

$$\mathbf{E}^\varepsilon(\mathbf{x},t) = -\nabla_{\mathbf{x}}\phi^\varepsilon, \qquad -\Delta_{\mathbf{x}}\phi^\varepsilon = \int_{\mathbb{R}^2} f^\varepsilon d\mathbf{v} - n_i, \tag{1.4}$$

$$f^\varepsilon(\mathbf{x},\mathbf{v},t=0) = f_0(\mathbf{x},\mathbf{v}), \tag{1.5}$$

where ϕ^ε is the electric potential and n_i is the background ion density. The system (1.3)-(1.4) is a first step towards a six dimensional model which can be used for the study of plasma under the influence of a strong magnetic field. The unknown $f^\varepsilon(\mathbf{x},\mathbf{v},t)$ represents the distribution of electrons in phase space at time t and thus, the system (1.3)-(1.4) describes the particle dynamics under the additional effect of the self-consistent electric field. The difficulty is that the large magnetic field, expressed by the $\mathbf{v}^\perp/\varepsilon$ term, introduces a new time scale, the rotation of particles around the magnetic field line, which is very small with respect to that of the electric field evolution. We are thus faced with a multi-scale problem whose numerical solution by standard methods requires heavy computational efforts.

We will also test our scheme when an external electric field in (1.1) is given by

$$\Xi^\varepsilon(\mathbf{x},t) = \begin{pmatrix} 2x_1+x_2 \\ x_1+2x_2 \end{pmatrix}, \tag{1.6}$$

which is the gradient of the potential $\varphi(x_1,x_2) = x_1^2 + x_1x_2 + x_2^2$. The reason for this particular case is twofold. First, we are able to write down the analytic solution to system (1.1)-(1.2) which thus leads to the capabilities of a real error computation and of testing our algorithm's main (second step) approximation alone. Second, we can write analytically the slow manifold (see Section 5), an important issue when testing the scheme for