Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2018-0208

Vol. **26**, No. 3, pp. 838-854
September 2019

Optimal Error Estimates of a Linearized Crank-Nicolson Galerkin FEM for the Kuramoto-Tsuzuki Equations

Dongfang Li[1,2], Waixiang Cao[3,*], Chengjian Zhang[1,2] and Zhimin Zhang[4,5]

[1] *School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China.*

[2] *Hubei Key Laboratory of Engineering Modeling and Scientific Computing, Huazhong University of Science and Technology, Wuhan 430074, China.*

[3] *School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China.*

[4] *Beijing Computational Science Research Center, Beijing 100193, China.*

[5] *Department of Mathematics, Wayne State University, Detroit, MI 48202, USA.*

Received 15 August 2018; Accepted (in revised version) 29 October 2018

Abstract. This paper is concerned with unconditionally optimal error estimate of the linearized Galerkin finite element method for solving the two-dimensional and three-dimensional Kuramoto-Tsuzuki equations, while the classical analysis for these nonlinear problems always requires certain time-step restrictions dependent on the spatial mesh size. The key to our analysis is to obtain the boundedness of the numerical approximation in the maximum norm, by using error estimates in certain norms in the different time level, the corresponding Sobolev embedding theorem, and the inverse inequality. Numerical examples in both 2D and 3D nonlinear problems are given to confirm our theoretical results.

AMS subject classifications: 65N30, 65N12, 65N15, 35B45

Key words: Unconditionally optimal error estimates, linearized Galerkin finite element method, Kuramoto-Tsuzuki equation, high-dimensional nonlinear problems.

1 Introduction

In this paper, we consider the optimal error estimates of linearized finite element method (FEM) for solving the following multi-dimensional Kuramoto-Tsuzuki equation

*Corresponding author. *Email addresses:* `dfli@hust.edu.cn` (D. Li), `caowx@bnu.edu.cn` (W. Cao), `cjzhang@hust.edu.cn` (C. Zhang), `zzhang@math.wayne.edu` (Z. Zhang)

$$u_t=(1+\mathrm{i}c_1)\Delta u+u-(1+\mathrm{i}c_2)|u|^2u, \quad x\in\Omega, \quad 0<t\leq T, \tag{1.1}$$
$$u(x,0)=u_0(x), \quad x\in\Omega, \tag{1.2}$$
$$u=0, \quad x\in\partial\Omega, \tag{1.3}$$

where $\mathrm{i}=\sqrt{-1}$, c_1 and c_2 are real constants, u is a complex unknown function defined in $\Omega\times[0,T]$, and $\Omega\subset\mathbb{R}^d$ for $d=2$ or 3 is bounded and convex polygon/polyhedron. The Kuramoto-Tsuzuki equation is widely used to describe the behavior of many two-component systems in a neighborhood of the bifurcation point [1,2]. It also can be viewed as a special case of the complex Ginzburg-Landau equation, which is widely used to describe a huge number of phenomena from nonlinear waves to second-order phase transitions, from superconductivity and Bose-Einstein condensation to liquid crystals and strings in field theory [3].

Many efforts have been made to develop effective numerical methods and analysis for solving the one dimensional Kuramoto-Tsuzuki equation. For example, Tsertsvadze [6] developed an implicit finite difference scheme and proved that it was convergent with the rate of order $\mathcal{O}(h^{\frac{3}{2}})$ in the discrete L_2-norm under the constraint $\tau=h^{2+\delta}$ with constant $\delta>0$. Here and below, h and τ denote the spatial and temporal mesh size, respectively. Sun [7] further proved that Tsertsvadze's scheme converges at the rate of $\mathcal{O}(\tau^2+h^2)$ by removing the constraint for one dimensional problem. Omrani [8] investigated a second-order convergent linearized Galerkin approximation to the one dimensional Kuramoto-Tsuzuki equation. Wang et al. [9] proposed a nonlinear finite difference scheme with the rate of order $\mathcal{O}(\tau^2+h^2)$. More results about numerical analysis for one dimensional problem can be found in the papers [10–14].

The error estimates of numerical schemes for high-dimensional nonlinear problems are usually derived under certain time-step restrictions dependent on the spatial mesh size. One of the important reasons is that one may apply the following inverse inequality to obtain the boundedness of $\|U_h^n\|_{L^\infty}$, i.e.,

$$\begin{aligned}\|U_h^n\|_{L^\infty}&\leq\|R_hu^n\|_{L^\infty}+\|R_hu^n-U_h^n\|_{L^\infty}\\&\leq\|R_hu^n\|_{L^\infty}+Ch^{-d/2}\|R_hu^n-U_h^n\|_{L^2}\\&\leq\|R_hu^n\|_{L^\infty}+Ch^{-d/2}(\tau^p+h^{r+1}),\end{aligned} \tag{1.4}$$

where R_h is the classical projection operator, p and $r+1$ are the orders of convergence in both temporal and spatial directions, U_h^n and u^n are the numerical and exact solutions. Clearly, a time-step condition $\tau=\mathcal{O}(h^{\frac{d}{2p}})$ is needed in (1.4). For the error estimates under such time-step restrictions, we refer to [15–21] for an incomplete list of references. If the similar restrictions are required, one may apply an unnecessarily small time step in actual application. Then it may make the computation more time-consuming.

Optimal error estimates without the above mentioned time-step restrictions is so-called unconditional convergence. Recently, Li and Sun [22, 23] proposed a new ap-