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Numerical Solution of an Inverse Obstacle Scattering Problem for Elastic Waves via the Helmholtz Decomposition

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Abstract. Consider an inverse obstacle scattering problem in an open space which is filled with a homogeneous and isotropic elastic medium. The inverse problem is to determine the obstacle's surface from the measurement of the displacement on an artificial boundary enclosing the obstacle. In this paper, a new approach is proposed for numerical solution of the inverse problem. By introducing two scalar potential functions, the method uses the Helmholtz decomposition to split the displacement of the elastic wave equation into the compressional and shear waves, which satisfy a coupled boundary value problem of the Helmholtz equations. The domain derivative is studied for the coupled Helmholtz system. In particular, we show that the domain derivative of the potentials is the Helmholtz decomposition of the domain derivative of the displacement for the elastic wave equation. Numerical results are presented to demonstrate the effectiveness of the proposed method.

AMS subject classifications: 78A46, 65N21

Key words: The elastic wave equation, inverse obstacle scattering, domain derivative, the Helmholtz decomposition.

1 Introduction

As one of the fundamental problems in scattering theory, the obstacle scattering problem is concerned with the effect that an impenetrable medium on an incident field. If the total field is viewed as the sum of an incident field and a scattered field, the direct obstacle scattering problem is to determine the scattered field from the incident field and

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the governing equation for the wave motion; the inverse obstacle scattering problem is to determine the shape of the medium from the measurement of the scattered field. These problems have played essential roles in many scientific areas, such as radar and sonar, nondestructive testing, medical imaging, and geophysical exploration.

Driven by significant applications, the direct and inverse obstacle scattering problems have been widely studied by numerous researchers for all the three commonly encountered wave models, which include the Helmholtz equation (acoustic waves), the Maxwell equations (electromagnetic waves), and the Navier equation (elastic waves). The inverse obstacle scattering problems are challenging due to nonlinearity and ill-posedness. Computational approaches can be broadly classified into two types: optimization based iterative methods and imaging based direct methods [8, 10]. The former are named as quantitative methods while the latter are referred to as qualitative methods. The iterative methods require good initial guesses and are computationally expensive as a sequence of direct and adjoint problems need to be solved at each step of iterations. The direct methods require no a priori information on the obstacles and are computationally efficient, but the reconstructions may not be as accurate as those by using iterative methods.

For the optimization based methods, it is inevitable to calculate the domain or Fréchet derivatives when applying linearization procedures for these nonlinear problems. The domain or Fréchet derivatives characterize the variation of wave field with respect to perturbation of media such as the boundary of the obstacle. The domain derivatives have been studied by many authors for the inverse acoustic and electromagnetic obstacle scattering problems. In [32], Roger investigated the differentiability of the far-field pattern with respect to the obstacle's boundary and employed the Newton–Kantorovitch iterative method to solve the inverse obstacle scattering problem. The Fréchet derivatives of the scattering operators were studied in [12,15,19] by using the variational approaches and in [28,29] by using the boundary integral equation techniques for either the Dirichlet, Neumann, or impedance boundary condition. Recently, some related numerical results can be found in [33] on profile reconstruction for a periodic transmission problem from single-sided data.

The scattering problems for elastic waves have continuously attracted much attention by many researchers due to their significant applications in such areas as geophysics and seismology [1, 6, 21, 27]. Elastic waves are governed by the Navier equation which is complex due to the coexistence of compressional and shear waves that propagate at different speeds. In [11, 13], Hahner and Hsiao, and Elschner and Yamamoto considered the uniqueness of the inverse elastic scattering problem, separately. Various numerical methods can be found in [2, 17, 25]. The Fréchet differentiability of the boundary integral operators was studied in [7]. In [22, 23], Louër investigated domain derivatives of the inverse obstacle scattering problem for elastic waves by using the boundary integral equation method. The domain derivatives were considered in [24, 26] for the two- and three-dimensional inverse elastic obstacle scattering problems by using the variational method, and a frequency recursive method was developed to reconstruct the surface of the obstacle. Related numerical results can be found in [4, 5, 9, 31] on solving the inverse