Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2017-0261

Vol. **26**, No. 1, pp. 135-159
July 2019

A Nonlinear Finite Volume Element Method Satisfying Maximum Principle for Anisotropic Diffusion Problems on Arbitrary Triangular Meshes

Yanni Gao[1], Shuai Wang[1], Guangwei Yuan[1,2] and Xudeng Hang[1,*]

[1] *Institute of Applied Physics and Computational Mathematics, Fenghaodong Road, Haidian district, Beijing 100094, China.*
[2] *Laboratory of Computational Physics, P.O.Box 8009, Beijing, 100088, China.*

Received 25 December 2017; Accepted (in revised version) 30 July 2018

Abstract. A nonlinear finite volume element scheme for anisotropic diffusion problems on general triangular meshes is proposed. Starting with a standard linear conforming finite volume element approximation, a corrective term with respect to the flux jumps across element boundaries is added to make the scheme satisfy the discrete maximum principle. The new scheme is free of the anisotropic non-obtuse angle condition which is a severe restriction on the grids for problems with anisotropic diffusion. Moreover, this manipulation can nearly keep the same accuracy as the original scheme. We prove the existence of the numerical solution for this nonlinear scheme theoretically. Numerical results and a grid convergence study are presented for both continuous and discontinuous anisotropic diffusion problems.

AMS subject classifications: 65N08, 65N12, 65N15

Key words: Finite volume element method, nonlinear correction, discrete maximum principle, anisotropic diffusion.

1 Introduction

We are concerned with the numerical solution of the diffusion equation:

$$-\nabla\cdot(\Lambda\nabla u)=f \quad \text{in } \Omega, \tag{1.1}$$
$$u=g \quad \text{on } \partial\Omega, \tag{1.2}$$

where

*Corresponding author. *Email addresses:* `gaoyn10@163.com` (Y. Gao), `hang_xudeng@iapcm.ac.cn` (X. Hang), `yuan_guangwei@iapcm.ac.cn` (G. Yuan), `wang_shuai@iapcm.ac.cn` (S. Wang)

 ©2019 Global-Science Press

(a) Ω is an open bounded, convex connected polygonal domain in $\mathbb{R}^2$ with the boundary $\partial\Omega$;

(b) f is the source term, belonging to $L^2(\Omega)$;

(c) g is the Dirichlet boundary data defined on $\partial\Omega$;

(d) Λ is a symmetric tensor such that Λ is piecewise Lipschitz-continuous on Ω and the set of eigenvalues of Λ is included in $[\lambda_{\min},\lambda_{\max}]$ with $\lambda_{\min}>0$.

The boundary value problem (BVP) (1.1)-(1.2) becomes an anisotropic diffusion problem if eigenvalues of Λ are not all equal at least on a portion of Ω. This kind of problem is a model arising in various fields such as plasma physics [19, 33], petroleum reservoir simulation [17], and image processing [38]. As typical for diffusion problems, it satisfies the maximum principle

$$\min_{\mathbf{x}\in\Omega\cup\partial\Omega} u(\mathbf{x}) \geq \min_{\mathbf{x}\in\partial\Omega} g(\mathbf{x}) \tag{1.3}$$

provided that $f(\mathbf{x})\geq 0$ holds for all $\mathbf{x}\in\Omega$. When using a standard numerical method, such as a finite element, a finite difference, or a finite volume method, to solve this problem, spurious oscillations may occur. In order to avoid such spurious oscillations in the numerical solution, a common strategy is to develop numerical schemes guaranteeing the discrete counterpart of (1.3), i.e., the so-called discrete maximum principle (DMP), which are known to produce numerical solutions evading nonphysical local oscillations or preserving positivity.

Development of DMP satisfaction schemes for solving diffusion problems has attracted considerable interest in the past. By virtue of the convex combination of two linear flux approximation and the positivity-preserving interpolation of the auxiliary unknowns, various cell-centered finite volume (FV) schemes circumventing spurious oscillations have been developed. These schemes usually have approximately a second-order accuracy on severely distorted meshes in the highly anisotropic, and/or discontinuous case. However, their extensions to finite element (FE) methods are hard to succeed. We refer readers to [1, 16, 18, 28, 30, 34, 39, 40] and references therein for more details. In the framework of FE methods, the study of DMP-preserving schemes for anisotropic diffusion case is more difficult and relevant results are very limited. In [27], the authors derive an anisotropic non-obtuse angle condition in term of the M-matrix criteria, such that the linear FE scheme guarantees the DMP by employing a suitable mesh. On the other hand, separating the stiffness matrix resulting from the FE discretization into diffusive and anti-diffusive fluxes and limiting the anti-diffusion fluxes by proper limiters lead to the so-called algebraic flux correction scheme. In [23] and [24], the authors propose two types of limiters to make the corrected schemes local extremum diminishing. But they are linearity-preserving only on symmetric meshes, as pointed out in [2].

Finite volume element (FVE) method [12,13,30,31], also called co-volume method [15] or generalized difference method [29], is one of the main numerical methods for solving