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# An Extremum-Preserving Iterative Procedure for the Imperfect Interface Problem

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**Abstract.** In this paper we propose an extremum-preserving iterative procedure for the imperfect interface problem. This method is based on domain decomposition method. First we divide the domain into two sub-domains by the interface, then we alternately solve the sub-domain problems with Robin boundary condition. We prove that the iterative method is convergent and the iterative procedure is extremum-preserving at PDE level. At last, some numerical tests are carried out to demonstrate the convergence of the iterative method by using a special discrete method introduced on sub-domains.

**AMS subject classifications**: 65N30

**Key words**: Imperfect interface, domain decomposition, iterative methods, extremum-preserving.

## 1 Introduction

There are a lot of literature about elliptic interface problem and most of them pay attention to the traditional type [6–8, 10, 17, 27], in which both the jump of the solution and the drop of the normal flux across the interface are given functions. For the imperfect contact interface problem, the solution experiences a sudden drop across the interface and the drop is proportional to the continuous normal flux. This kind of interface structure makes it difficult to solve and the work about the numerical methods for it is few [9, 26, 28]. However, this kind of problem is widely used such as in heat transfer of composite media [5, 11, 23], heat transfer of building and so on [18, 24].

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For this kind of interface problem, the existence of the weak solution is proved in [9] by introducing a nonstandard variational formulation, what's more, a nonstandard Galerkin method is used to get the numerical solution for it. A special iterative method, which is based on the combination of the standard iterative methods with some penalization theory, is proposed in [28] to solve the discretized problem derived in [9]. The nonstandard finite element method is used to the calculation in fibrous composites with an interface thermal resistance in [23]. Recently, the work [26] proposed a non-traditional finite element method with non-body-fitting grids to solve the matrix coefficient elliptic equations with imperfect contact in two dimensions based on the work [7, 8].

Domain decomposition method is an efficient method to solve elliptic and parabolic problems [16, 22, 25]. The method can divide the problem into small ones, so different models and methods can be introduced to solve the problem in the sub-domains. What's more, domain decomposition is a natural approach to solve the interface problems and the elliptic equations with discontinuous coefficients [19–21].

It is difficult to use continuous Galerkin method directly on the global domain because of the special interface structure and variational formulation. Some special techniques should be introduced when the nonstandard FEM is used. In this paper, we propose a kind of iterative method for the second-order elliptic interface problem with imperfect contact. This method is based on domain decomposition method and the interface conditions can be immersed into the boundary conditions naturally on sub-domains. In the construction of the method we take the interface as a part of the boundary of the sub-domains, and alternately solve the sub-problems. Moreover, we prove that the iterative method is convergent and the procedure is extremum-preserving. Some numerical tests are carried out to demonstrate the convergence of the iterative method.

The iterative method we proposed in this work has the following properties:

- Convergence in theory: the method is convergent in $L^2$ norm with geometric rate;
- Extremum-preserving: The iterative procedure is an extremum-preserving one, which can avoid non-physical solution;
- Independent of discrete method: our method gives an iterative method for model problem at the PDE level, so it does not depend on the discrete method on sub-domains, this property is promising especially on distorted meshes [30]. However, because of the property of extremum-preserving, some positivity preserving and extremum-preserving method can be used to solve the sub-domain problems [2, 12–14] ;
- Easy to implement: the problem on sub-domains are standard mixed boundary value problems, so many efficient numerical methods can be used without any changes [29].

The rest of this paper is organized as follows. In the next section, we give the model problem and some notations. The algorithm is described and some theoretical analysis is given in Section 3. In Section 4, a standard discrete method is introduced on sub-domains