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# Approximation of the Mean Escape Time for a Tilted Periodic Potential

Tamra Heberling[1,*], Lisa Davis[2] and Tomas Gedeon[2]

[1] *Los Alamos National Laboratory, XTD-SS, Los Alamos, NM, 87545, USA.*
[2] *Department of Mathematical Sciences, Montana State University, Bozeman, MT, 59717-2400, USA.*

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**Abstract.** We present a formula approximating the mean escape time (MST) of a particle from a tilted multi-periodic potential well. The potential function consists of a weighted sum of a finite number of component functions, each of which is periodic. For this particular case, the least period of the potential function is a common period amongst all of its component functions. An approximation of the MST for the potential function is derived, and this approximation takes the form of a product of the MSTs for each of the individual periodic component functions. Our first example illustrates the computational advantages of using the approximation for model validation and parameter tuning in the context of the biological application of DNA transcription. We also use this formula to approximate the MST for an arbitrary tilted periodic potential by the product of MSTs of a finite number of its Fourier modes. Two examples using truncated Fourier series are presented and analyzed.

**AMS subject classifications**: 60G10, 35Q84, 60J60, 60J65, 60H35

**Key words**: Brownian Ratchet, multi-periodic potential, tilted periodic potential, Fokker-Plank equation.

## 1 Introduction

Brownian ratchets were first introduced in the early 1900s by Smoluchowski and later developed by Feynman [23, 51]. A Brownian ratchet is a system where periodic forcing coupled with Brownian motion can be harnessed and directed to do work, such as particle transport. Brownian ratchets are often modeled as particle transport through a periodic potential where the Brownian motion is incorporated as a thermal noise term

*Corresponding author. *Email addresses:* `theberling@lanl.gov` (T. Heberling), `lisa.davis@montana.edu` (L. Davis), `tgedeon@montana.edu` (T. Gedeon)

 

used to propel the particle from one potential well, over a potential wall, and into the next well. Since the introduction of Brownian ratchets, they have been used in several fields to analyze diffusive motion, typically in microscopic systems where noise is fundamental to movement. Some examples include motor proteins [5], intercellular transport [6], DNA transcription [1,3,4,15,26,27,32,49], Josephson junctions [12], ring-laser gyroscopes [13], and actin polymerization in neuroscience [41,52].

Motivation for this work comes from seeking a model of RNA polymerase (RNAP) translocation along a DNA strand. RNAP motion is often modeled as a Brownian ratchet, where the periodic structure of $U(x)$ is aligned with RNAP transcription of successive nucleotides and the tilt $Fx$ biases RNAP motion in one direction along the DNA strand. However, it is well-established in the literature that each RNAP experiences short pauses in its motion, where it stalls at some nucleotide location for a brief amount of time before resuming elongation. These so called *transcriptional pauses* occur at random locations with a mean frequency of about 1 per 100 nucleotides [39]. Therefore one particular model of RNAP motion on DNA with pauses may be a Brownian motion in tilted potential with $U(x)=U_1(x)+U_2(x)$, where $U_1(x,L)$ is periodic with period $L$ corresponding to 100 nucleotides and $U_2(x,L/100)$ has a least period of $L/100$.

We investigate a one-dimensional random walk generated by Gaussian white noise, $\xi(t)$, and influenced by a tilting force $F$. This stochastic model is formulated as

$$\frac{dx}{dt}=-\frac{d}{dx}(U(x)-Fx)+\sqrt{2D}\xi(t), \qquad t>0, \tag{1.1}$$

where $x(t)$ is the one dimensional position at time $t$, and $D$ is the noise intensity. Often the parameter $D$ is given by the Einstein relation [18],

$$D=\frac{k_\beta T}{\gamma}, \tag{1.2}$$

where $k_\beta$ is the Boltzmann constant, $T$ is temperature, and $\gamma$ is friction. The tilting force is included in the model by subtracting a tilting term, $Fx$, from the periodic potential denoted as $U(x)$ in the equation above, see [43]. If $F<0$, the force pulls particles to the left and if $F>0$, the force pulls particles to the right. Using this tilting term, the effective potential in Eq. (1.1) is

$$V(x)=U(x)-Fx, \tag{1.3}$$

where $U(x)$ is periodic with period $L$. One way to approach this problem is as Kramers problem in periodic potentials, to find the rate that a particle escapes from a potential well [31]. Kramers problem is an important area of research in this field [8,9,11,38,46]. Several exact solutions have been proposed and analyzed [20–22,36,37].

Our main motivation for interest in the mean escape time (MST) of a Brownian particle over a wall comes from the fact that the characteristics of the long range behavior of the particle (i.e. over many periods and a long time) can be computed by decomposing the motion into two components. One is the motion on the spatial scale of a single