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The Weak Galerkin Method for Linear Hyperbolic Equation

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Abstract. The linear hyperbolic equation is of great interest in many branches of physics and industry. In this paper, we use the weak Galerkin method to solve the linear hyperbolic equation. Since the weak Galerkin finite element space consists of discontinuous polynomials, the discontinuous feature of the equation can be maintained. The optimal error estimates are proved. Some numerical experiments are provided to verify the efficiency of the method.

AMS subject classifications: 65N30, 65N15, 65N25, 35B45, 35J50, 35J35

Key words: Weak Galerkin finite element method, linear hyperbolic equation, error estimate.

1 Introduction

The linear hyperbolic equation arises in many branches of physics, including acoustics and fluid mechanics. For example, in the computational fluid dynamics the Lagrangian grids are usually employed, and the physical quantities, like density, velocity and pressure, extend from one medium to another medium through the interface. A linear hyperbolic equation, which is also called the eikonal equation, needs to be solved to verify the physical quantities on the ghost element near the interface. As to the derivation and more applications of the linear hyperbolic equation, readers are referred to [7] and the references therein.

Many numerical methods have been applied to the linear hyperbolic equation, such as the finite difference method [17], the finite element method [18], and the finite volume method [1,6]. A key issue of the numerical simulation of the linear hyperbolic equation

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is the approximation near the shock. The exact solution of the linear hyperbolic equation may be discontinuous, and it is a challenge for the numerical scheme to avoid oscillation around the discontinuity. In this aspect, the discontinuous Galerkin method [4] is a competitive candidate. The optimal order estimate of the discontinuous Galerkin method is discussed in [11]. The superconvergence phenomenon of the discontinuous Galerkin method is also studied, the $k+2$ order superconvergence and $2k+1$ order superconvergence are investigated in [24] and [2], respectively. There are also many other schemes for the linear hyperbolic equation, such as SUPG [8,9] and least squares method [5,10].

Recently, a numerical method called the weak Galerkin (WG) finite element method is proposed for solving PDEs. The weak Galerkin method has been introduced and analyzed in [20] for the second order elliptic equations. The main idea of the WG method is to use totally discontinuous polynomials as basis functions, and replace the classical derivative operators by specifically defined weak derivative operators in the numerical scheme. It has been applied to a variety of PDEs, including the second order elliptic equation [3,12,21], the biharmonic equation [14,15,27], the Stokes equation [16,22,26], the Brinkman equation [13,23,25] and the linear elasticity equation [19], etc. The weak Galerkin method employs discontinuous polynomials in the finite element space, which can help describe the discontinuity of the solution.

In the computational fluid dynamics, the mesh grid is usually polytopal and unstructured. The WG method can solve this kind of problems efficiently since it utilizes discontinuous elements and suits for polytopal meshes. The numerical simulation of the linear hyperbolic equation is also an important issue in computational fluid dynamics, and we are interested in solving this problem by the WG method.

We consider the linear hyperbolic equation that seeks an unknown function u satisfying

$$\boldsymbol{\beta}\cdot\nabla u+cu=f, \quad \text{in } \Omega, \tag{1.1}$$

$$u=g, \quad \text{on } \Gamma_-, \tag{1.2}$$

where Ω is a polytopal domain in $\mathbb{R}^d$ (polygonal or polyhedral domain for $d=2,3$), the coefficients $\boldsymbol{\beta}$ and c are non-negative functions, and

$$\Gamma_-=\{\mathbf{x}\in\partial\Omega,\boldsymbol{\beta}\cdot\mathbf{n}\leq 0 \text{ at } \mathbf{x}\}.$$

For the simplicity of analysis, we suppose $\boldsymbol{\beta}$ and c are piecewise constants.

In this paper, we apply the WG method to the linear hyperbolic equation, and give the corresponding estimates.

The rest of paper is structured as follows. In Section 2, we introduce some notations, definitions, and the WG scheme. In Section 3 we derive the error equations for the WG approximations and we give the error estimates. Some numerical experiments are presented in Section 4.