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High Order Accurate Direct Arbitrary-Lagrangian-Eulerian ADER-MOOD Finite Volume Schemes for Non-Conservative Hyperbolic Systems with Stiff Source Terms

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Abstract. In this paper we present a 2D/3D high order accurate finite volume scheme in the context of direct Arbitrary-Lagrangian-Eulerian algorithms for general hyperbolic systems of partial differential equations with non-conservative products and stiff source terms. This scheme is constructed with a single stencil polynomial reconstruction operator, a one-step space-time ADER integration which is suitably designed for dealing even with stiff sources, a nodal solver with relaxation to determine the mesh motion, a path-conservative integration technique for the treatment of non-conservative products and an *a posteriori* stabilization procedure derived from the so-called Multi-dimensional Optimal Order Detection (MOOD) paradigm. In this work we consider the seven equation Baer-Nunziato model of compressible multi-phase flows as a representative model involving non-conservative products as well as relaxation source terms which are allowed to become stiff. The new scheme is validated against a set of test cases on 2D/3D unstructured moving meshes on parallel machines and the high order of accuracy achieved by the method is demonstrated by performing a numerical convergence study. Classical Riemann problems and explosion problems with exact solutions are simulated in 2D and 3D. The overall numerical code is also profiled to provide an estimate of the computational cost required by each component of the whole algorithm.

AMS subject classifications: 65M08, 65M60

Key words: Direct Arbitrary-Lagrangian-Eulerian, a posteriori MOOD stabilization, Baer-Nunziato model, stiff source terms, non-conservative products, unstructured mesh, ADER, high order of accuracy in space and time, high performance computing (HPC), hyperbolic conservation laws.

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1 Introduction

In this work we pursue the development of the direct cell-centered Arbitrary-Lagrangian-Eulerian (ALE) ADER algorithm [10] supplemented with *a posteriori* Multi-dimensional Optimal Order Detection (MOOD) stabilization technique. In [13] the ALE-ADER-MOOD numerical scheme has been solely tested and validated on the hydrodynamics system of conservation laws in multiple space dimensions on unstructured meshes. The hydrodynamics equations have been used as a first non-trivial hyperbolic system generating complex physical features (vorticity, shocks, contacts, wave interactions, etc.) which might be difficult to capture with a numerical scheme. We have shown that the ALE-ADER-MOOD approach can accurately solve such a system in 2D and 3D on unstructured moving grids.

Nonetheless, the Euler equations of compressible gas dynamics are relatively simple compared to more advanced models used in different fields of computational physics. Unfortunately the complexity of these models immutably leads to more equations to be solved as well as to more demanding terms, such as stiff sources, non-conservative products and/or constraints (divergence free, positivity). Dealing with such systems is an interesting challenge for high-order numerical schemes on moving unstructured meshes.

In this work we consider the seven-equation Baer-Nunziato (BN) system [2] that models compressible two-phase flows. The BN model consists in the combination of two systems of compressible Euler equations, one for each phase, coupled together via non-conservative products and relaxation sources. Exact numerical solutions, in ideal situations, can also be derived, which help the validation of our numerical method. The Baer-Nunziato system is used in this paper as a representative example of more complex physical systems of PDEs encountered in other fields of physics, hence requiring efficient, accurate and robust algorithms to be solved numerically.

Our aim is to show that the *a posteriori* MOOD stabilization technique developed in [13, 16, 24, 25, 45] can deal with such complex balance laws for which the occurrence of non-conservative and/or (stiff) source terms is part of modeling, and, as such, must be properly solved. The general framework for our study originates from [10], where the *a priori* WENO technique is then replaced by an *a posteriori* MOOD technique [13]. Within the *a posteriori* MOOD context several difficult properties are more easily fulfilled such as the positivity preservation of some physical quantities (density, internal energy and pressure). This substitution must be adapted to the presence of non-conservative products and stiff sources.

The rest of this paper is organized as follows. The second section briefly presents the direct high order accurate unlimited ADER Arbitrary-Lagrangian-Eulerian (ALE) scheme that was previously designed in [10], while in Section 3 the *a posteriori* MOOD technique is fully described, including the detection criteria which are properly selected and chosen for the Baer-Nunziato model. Decrementing technique as well as implementation issues and developer choices (cascade of schemes, parachute bullet-proof scheme) are discussed there. In Section 4 we gather the numerical results for a set of test cases