A New Coupled Complex Boundary Method for Bioluminescence Tomography

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Abstract. In this paper, we introduce and study a new method for solving inverse source problems, through a working model that arises in bioluminescence tomography (BLT). In the BLT problem, one constructs quantitatively the bioluminescence source distribution inside a small animal from optical signals detected on the animal’s body surface. The BLT problem possesses strong ill-posedness and often the Tikhonov regularization is used to obtain stable approximate solutions. In conventional Tikhonov regularization, it is crucial to choose a proper regularization parameter for trade off between the accuracy and stability of approximate solutions. The new method is based on a combination of the boundary condition and the boundary measurement in a parameter-dependent single complex Robin boundary condition, followed by the Tikhonov regularization. By properly adjusting the parameter in the Robin boundary condition, we achieve two important properties for our new method: first, the regularized solutions are uniformly stable with respect to the regularization parameter so that the regularization parameter can be chosen based solely on the consideration of the solution accuracy; second, the convergence order of the regularized solutions reaches one with respect to the noise level. Then, the finite element method is used to compute numerical solutions and a new finite element error estimate is derived for discrete solutions. These results improve related results found in the existing literature. Several numerical examples are provided to illustrate the theoretical results.

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Key words: Bioluminescence tomography, Tikhonov regularization, convergence rate, finite element methods, error estimate.

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1 Introduction

Bioluminescence tomography (BLT) is a new molecular imaging modality and has shown its potential in monitoring non-invasively physiological and pathological processes \emph{in vivo} at the cellular and molecular level. It is particularly attractive for \emph{in vivo} applications because no external excitation source is needed and thus background noise is low while sensitivity is high [21]. In the BLT problem, one reconstructs quantitatively the bioluminescence source distribution inside a small animal from optical signals detected on the animal’s body surface. Let $\Omega \subset \mathbb{R}^d$ ($d \leq 3$: space dimension) be an open bounded set with boundary $\Gamma = \partial \Omega$. Then without going into detail, we state the BLT problem as follows.

**Problem 1.1.** Find a source function $p$ inside $\Omega$ so that the solution $u$ of the forward (real) Robin boundary value problem (BVP)

\[
\begin{aligned}
-\text{div}(D \nabla u) + \mu_a u &= p \quad \text{in } \Omega, \\
\vphantom{\mu_a} u + 2AD\frac{\partial u}{\partial n} &= g^- \quad \text{on } \Gamma
\end{aligned}
\]

(1.1)

satisfies the outgoing flux density on the boundary:

\[
g = -D\frac{\partial u}{\partial n} \quad \text{on } \Gamma_0. \tag{1.2}
\]

Here $D = [3(\mu_a + \mu')]^{-1}$ is the diffusion coefficient with $\mu_a$ and $\mu'$ being known as the absorption and reduced scattering parameters; $\partial / \partial n$ stands for the outward normal derivative; $g^-$ is an incoming flux on $\Gamma$ and it vanishes when the imaging is implemented in a dark environment; $\Gamma_0 \subset \Gamma$ is the part of the boundary for measurement; $A = A(x) = (1 + R(x)) / ((1 - R(x)))$ with $R(x) \approx -1.4399\gamma(x)^{-2} + 0.7099\gamma(x)^{-1} + 0.6681 + 0.0636\gamma(x)$ and $\gamma(x)$ being the refractive index of the medium at $x \in \Gamma$. In what follows, we restrict ourselves to the case where $g^- \equiv 0$ and $\Gamma_0 = \Gamma$.

Inverse source problems with only one measurement on the boundary do not have a unique solution. In the context of the BLT problem, one cannot distinguish between a strong source over a small region and a weak source over a large region. For instance, let $\Omega$ be the unit circle centered at the origin, $\mu_a = 0.04$, $\mu' = 1.5$, and $A = 3.2$ with refractive index $\gamma = 1.3924$. We assign two different source functions: a strong small source function $p_1 = 4$ in a circle centered $(0.5,0)$ with radius 0.2 and a weak big source function $p_2 = 1$ in a circle centered $(0.5,0)$ with radius 0.4. Although the solutions $u_1$ and $u_2$ of (1.1), corresponding to $p_1$ and $p_2$ respectively, differ greatly in $\Omega$, they have almost the same outgoing flux density $g$ on the boundary, as is shown in Fig. 1. This agrees with the theoretical result about the solution non-uniqueness presented in [12]. One can have better identification with more a priori information about the source function $p$. One of the a priori information is a permissible region $\Omega_0 \subset \Omega$ of the optical source distribution. In this case, the first equation of (1.1) is replaced by

\[ -\text{div}(D \nabla u) + \mu_a u = p\chi_{\Omega_0} \quad \text{in } \Omega, \]