Commun. Comput. Phys.
doi: 10.4208/cicp.180914.121214a

Vol. **18**, No. 1, pp. 65-90
July 2015

Discrete Maximum Principle for the Weak Galerkin Method for Anisotropic Diffusion Problems

Weizhang Huang[1] and Yanqiu Wang[2,*]

[1] *Department of Mathematics, The University of Kansas, Lawrence, KS 66045, U.S.A.*
[2] *Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, U.S.A.*

Received 18 September 2014; Accepted 12 December 2014

Abstract. A weak Galerkin discretization of the boundary value problem of a general anisotropic diffusion problem is studied for preservation of the maximum principle. It is shown that the direct application of the M-matrix theory to the stiffness matrix of the weak Galerkin discretization leads to a strong mesh condition requiring all of the mesh dihedral angles to be strictly acute (a constant-order away from 90 degrees). To avoid this difficulty, a reduced system is considered and shown to satisfy the discrete maximum principle under weaker mesh conditions. The discrete maximum principle is then established for the full weak Galerkin approximation using the relations between the degrees of freedom located on elements and edges. Sufficient mesh conditions for both piecewise constant and general anisotropic diffusion matrices are obtained. These conditions provide a guideline for practical mesh generation for preservation of the maximum principle. Numerical examples are presented.

AMS subject classifications: 65N30, 65N50

Key words: Discrete maximum principle, weak Galerkin method, anisotropic diffusion.

1 Introduction

We are concerned with the discrete maximum principle for a weak Galerkin discretization of the boundary value problem (BVP) of a two-dimensional diffusion problem,

$$\begin{cases} -\nabla\cdot(\mathcal{A}\nabla u)=f, & \text{in } \Omega, \\ u=g, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^2$ is a polygonal domain, f and g are given functions, and $\mathcal{A}$ is a symmetric and uniformly positive definite diffusion matrix defined on Ω. The problem is isotropic

*Corresponding author. *Email addresses:* `whuang@ku.edu` (W. Huang), `yanqiu.wang@okstate.edu` (Y. Wang)

when the diffusion matrix takes the form $\mathcal{A}=\alpha(\mathbf{x})I$ for some scalar function $\alpha(\mathbf{x})$ and anisotropic otherwise. In this work we are interested in the anisotropic situation. It is known (e.g., see Evans [10]) that the classical solution of (1.1) satisfies the maximum principle,

$$f \le 0 \text{ in } \Omega \quad \Longrightarrow \quad \max_{\mathbf{x}\in\Omega\cup\partial\Omega} u(\mathbf{x}) = \max_{\mathbf{x}\in\partial\Omega} u(\mathbf{x}). \tag{1.2}$$

It is theoretically and practically important to investigate if a numerical approximation to (1.1) preserves such a property. Indeed, preservation of the maximum principle has attracted considerable attention from researchers; e.g., see [4,6,8,9,14–18,20–22,24–26,31–35,39–42]. For example, it is shown by Ciarlet and Raviart [8] and Brandts et al. [4] that P1 conforming finite element (FE) solutions to isotropic diffusion problems satisfy a discrete maximum principle (DMP) if all of the mesh elements have nonobtuse dihedral angles. This nonobtuse angle condition can be replaced in two dimensions by a weaker condition (the Delaunay condition) [34] requiring the sum of any pair of angles facing a common interior edge to be less than or equal to π. For anisotropic diffusion problems, Drăgănescu et al. [9] show that the nonobtuse angle condition fails to guarantee the satisfaction of DMP for a P1 conforming FE approximation. Various techniques, including local matrix modification [17, 24], mesh optimization [26], and mesh adaptation [22], have been proposed to reduce spurious oscillations. More recently, it is shown by Li and Huang [20] that P1 conforming FE solutions to anisotropic diffusion problems can be guaranteed to satisfy DMP if the mesh satisfies an anisotropic nonobtuse angle condition where mesh dihedral angles are measured in the metric specified by $\mathcal{A}^{-1}$ instead of the Euclidean metric. The result is extended to two dimensional problems [14], problems with convection and reaction terms [25], and time dependent problems [21]. It is emphasized that while DMP has been well studied for conforming FE discretizations, it is less explored for nonconforming or mixed/mixed-hybrid FE methods. Noticeably, DMP has been proven by Gu [11] for a nonconforming FE discretization and by Hoteit et al. [13] and Vohralík and Wohlmuth [36] for mixed/mixed-hybrid FE discetizations. However, their results focus on isotropic diffusion problems. Little is known about those discretizations for anisotropic diffusion problems.

The objective of this paper is to investigate the preservation of the maximum principle by a weak Galerkin approximation of BVP (1.1) with a general anisotropic diffusion matrix $\mathcal{A}$. The weak Galerkin method, recently introduced by Wang and Ye [38], is a FE method which uses a discontinuous FE space and approximates derivatives with weakly defined ones on functions with discontinuity. It can be easy to implement for meshes containing arbitrary polygonal/polyhedral elements [27,29,37,38]. The method has been successfully applied to various model problems [28, 29], and its optimal order convergence has been established for second order elliptic equations [29, 37, 38]. On the other hand, the weak Galerkin method has not been studied in the aspect of preserving the maximum principle. Such studies are useful in practice to avoid unphysical numerical solutions. They are also beneficial in theory since they provide in-depth understandings of the newly developed weak Galerkin method. It should be pointed out that such