Computing the Smallest Eigenvalue of Large Ill-Conditioned Hankel Matrices

Niall Emmart[1,*], Yang Chen[2] and Charles C. Weems[1]

[1] *College of Information and Computer Science, University of Massachusetts, Amherst, MA 01002, USA.*
[2] *Department of Mathematics, University of Macau, Macau, China.*

Abstract. This paper presents a parallel algorithm for finding the smallest eigenvalue of a family of Hankel matrices that are ill-conditioned. Such matrices arise in random matrix theory and require the use of extremely high precision arithmetic. Surprisingly, we find that a group of commonly-used approaches that are designed for high efficiency are actually less efficient than a direct approach for this class of matrices. We then develop a parallel implementation of the algorithm that takes into account the unusually high cost of individual arithmetic operations. Our approach combines message passing and shared memory, achieving near-perfect scalability and high tolerance for network latency. We are thus able to find solutions for much larger matrices than previously possible, with the potential for extending this work to systems with greater levels of parallelism. The contributions of this work are in three areas: determination that a direct algorithm based on the secant method is more effective when extreme fixed-point precision is required than are the algorithms more typically used in parallel floating-point computations; the particular mix of optimizations required for extreme precision large matrix operations on a modern multi-core cluster, and the numerical results themselves.

1 Introduction

In the majority of standard matrix computations, the matrix elements come from measurements of one kind or another. Accurate measurements might have 6-15 significant

---

Corresponding author. Email addresses: `nemmart@yrrid.com` (N. Emmart), `yayangchen@umac.mo` (Y. Chen), `weems@cs.umass.edu` (C. C. Weems)

digits. In most cases, computations with double precision arithmetic and the right pivoting strategy produce results that are valid to the precision allowed by the initial measurements. The round-off errors introduced during the computation are generally much smaller than the uncertainty in the original measurements. In this paper, we examine a class of ill-conditioned matrix problems from random matrix theory, whose elements come not from measurements but are given by an explicit formula that can be evaluated to arbitrary precision. Unlike standard problems, to perform accurate computations, the initial matrix must be evaluated to thousands of digits of precision and the intermediate computations must be performed with tens of thousands of digits of precision. The extreme precision requires enormous computing resources, and presents novel challenges and trade offs.

In particular, we study the problem of quickly and efficiently computing the smallest eigenvalue of a family of ill-conditioned Hankel matrices. Recall that Hankel or moments matrices, denoted as A, are obtained through (positive) weight functions $w(x)$ supported on $\mathbb{R}$ or subsets of $\mathbb{R}$, and are defined by

$$\mathtt{A}_{i,j} = \mu_{i+j} := \int_a^b x^{i+j} w(x) dx, \quad (i,j=0,1,2,\cdots).$$

These matrices are symmetric, and generate a positive definite quadratic form. For the problem at hand, we consider

$$w(x) := e^{-x^\beta}, \quad 0 \le x < \infty, \quad \beta > 0,$$

and the moments are given by the formula:

$$\mu_j = \int_0^\infty x^j e^{-x^\beta} dx = \frac{1}{\beta} \Gamma\left(\frac{1+j}{\beta}\right), \quad (j=0,1,2,\cdots).$$

We denote the order N Hankel matrix by $\mathtt{A}_N = (A_{i,j})_{0 \le i,j \le N-1}$. $\Gamma(z)$ is the gamma function and for complex numbers with a positive real part, gamma is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

The following are two example matrixes for $N=4$, $\beta=1$ and $N=4$, $\beta=\frac{7}{4}$:

$$\begin{bmatrix} 0! & 1! & 2! & 3! \\ 1! & 2! & 3! & 4! \\ 2! & 3! & 4! & 5! \\ 3! & 4! & 5! & 6! \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{4}{7}\Gamma(\frac{4}{7}) & \frac{4}{7}\Gamma(\frac{8}{7}) & \frac{4}{7}\Gamma(\frac{12}{7}) & \frac{4}{7}\Gamma(\frac{16}{7}) \\ \frac{4}{7}\Gamma(\frac{8}{7}) & \frac{4}{7}\Gamma(\frac{12}{7}) & \frac{4}{7}\Gamma(\frac{16}{7}) & \frac{4}{7}\Gamma(\frac{20}{7}) \\ \frac{4}{7}\Gamma(\frac{12}{7}) & \frac{4}{7}\Gamma(\frac{16}{7}) & \frac{4}{7}\Gamma(\frac{20}{7}) & \frac{4}{7}\Gamma(\frac{24}{7}) \\ \frac{4}{7}\Gamma(\frac{16}{7}) & \frac{4}{7}\Gamma(\frac{20}{7}) & \frac{4}{7}\Gamma(\frac{24}{7}) & \frac{4}{7}\Gamma(\frac{28}{7}) \end{bmatrix}.$$

In the numerical section of [9], Chen and Lawrence used the Jacobi rotation method to compute the smallest eigenvalues for matrices up to size 300 by 300. For the 300 by