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Analysis of a Two-Level Algorithm for HDG Methods for Diffusion Problems

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Abstract. This paper analyzes an abstract two-level algorithm for hybridizable discontinuous Galerkin (HDG) methods in a unified fashion. We use an extended version of the Xu-Zikatanov (X-Z) identity to derive a sharp estimate of the convergence rate of the algorithm, and show that the theoretical results also are applied to weak Galerkin (WG) methods. The main features of our analysis are twofold: one is that we only need the minimal regularity of the model problem; the other is that we do not require the triangulations to be quasi-uniform. Numerical experiments are provided to confirm the theoretical results.

AMS subject classifications: 52B10, 65D18, 68U05, 68U07

Key words: two-level algorithm, hybridizable discontinuous Galerkin method, weak Galerkin method, multigrid, X-Z identity.

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1 Introduction

The Hybridizable Discontinuous Galerkin (HDG) framework, proposed in [17] (2009) for second order elliptic problems, provides a unifying strategy for hybridization of finite element methods. The unifying framework includes as particular cases hybridized versions of mixed methods [2, 7, 15], the continuous Galerkin (CG) method [19], and a wide class of hybridizable discontinuous Galerkin (HDG) methods. Here hybridization denotes the process to rewrite a finite element method as a hybrid version. It should be pointed out that the Raviart-Thomas (RT) [32] and Brezzi-Douglas-Marini (BDM) mixed methods were first shown in [2,7] to have equivalent hybridized versions, and an overview of some hybridization techniques was presented in [16]. In the so-called HDG methods following the HDG framework, the constraint of function continuity on the inter-element boundaries is relaxed by introducing Lagrange multipliers defined on the inter-element

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boundaries, thus allowing for piecewise-independent approximation to the potential or flux solution. By local elimination of the unknowns defined in the interior of elements, the HDG methods finally lead to symmetric and positive definite (SPD) systems where the unknowns are only the globally coupled degrees of freedom describing the Lagrange multipliers. We refer to [13, 18, 24] for the convergence analysis of several HDG methods for the second order elliptic problems.

Closely related to the HDG framework is the weak Galerkin (WG) finite element method [29–31,34] pioneered by Wang and Ye [34]. The WG method is designed by using a weakly defined gradient operator over functions with discontinuity, and then allows the use of totally discontinuous piecewise polynomials in the finite element procedure. By introducing the discrete weak gradient as an independent variable, as shown in [23], the WG method can be rewritten as some HDG version when the diffusion-dispersion tensor in the corresponding second order elliptic equation is a piecewise-constant matrix.

It is well-known that the design of fast solvers is a key component to numerically solving partial differential equations. For the HDG methods as well as the WG methods, so far there are only limited literature concerning this issue. In [22], Gopalakrishnan and Tang analyzed a $\mathcal{V}$-cycle multigrid algorithm for two type of HDG methods for the Poisson problem with full elliptic regularity. By following the same idea, Cockburn et al. [14] presented the first convergence study of a nonnested $\mathcal{V}$-cycle multigrid algorithm for one type of HDG method for diffusion equations without full elliptic regularity. Chen et al. [11] constructed two auxiliary space multigrid preconditioners for two types of WG methods for the diffusion equations. In [23], Li and Xie proposed a two-level algorithm for two types of WG methods without full elliptic regularity, and, in [25], they analyzed an optimal BPX preconditioner for a large class of nonstandard finite element methods for the diffusion equations, including the hybridized Raviart-Thomas and Brezzi-Douglas-Marini mixed element methods, the HDG methods, the WG methods, and the nonconforming Crouzeix-Raviart element method.

In this paper, we shall propose and analyze an abstract two-level algorithm for the SPD systems arising from the HDG methods for the following diffusion model:

$$\begin{cases} -\mathrm{div}(\boldsymbol{a}\boldsymbol{\nabla} u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^d$ $(d = 2,3)$ is assumed to be a bounded polyhedral domain, the diffusion-dispersion tensor $\boldsymbol{a} \in [L^\infty(\Omega)]^{d\times d}$ is a SPD matrix and $f \in L^2(\Omega)$. In the two-level algorithm, the H^1-conforming piecewise linear finite element space is used as the auxiliary space. The main tool of our analysis is an extended version of the Xu-Zikatanov (X-Z) identity [36]. The main features of our work are as follows:

- We only need the minimal regularity of the model problem (1.1) in the sense that the regularity estimate
$$\|u\|_{1+\alpha,\Omega} \leqslant C_\Omega \|f\|_{\alpha-1,\Omega} \tag{1.2}$$