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Efficient Variable-Coefficient Finite-Volume Stokes Solvers

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Abstract. We investigate several robust preconditioners for solving the saddle-point linear systems that arise from spatial discretization of unsteady and steady variable-coefficient Stokes equations on a uniform staggered grid. Building on the success of using the classical projection method as a preconditioner for the coupled velocity-pressure system [*B. E. Griffith, J. Comp. Phys., 228 (2009), pp. 7565–7595*], as well as established techniques for steady and unsteady Stokes flow in the finite-element literature, we construct preconditioners that employ independent generalized Helmholtz and Poisson solvers for the velocity and pressure subproblems. We demonstrate that only a single cycle of a standard geometric multigrid algorithm serves as an effective inexact solver for each of these subproblems. Contrary to traditional wisdom, we find that the Stokes problem can be solved nearly as efficiently as the independent pressure and velocity subproblems, making the overall cost of solving the Stokes system comparable to the cost of classical projection or fractional step methods for incompressible flow, even for steady flow and in the presence of large density and viscosity contrasts. Two of the five preconditioners considered here are found to be robust to GMRES restarts and to increasing problem size, making them suitable for large-scale problems. Our work opens many possibilities for constructing novel unsplit temporal integrators for finite-volume spatial discretizations of the equations of low Mach and incompressible flow dynamics.

AMS subject classifications: 65F08, 65F10

Key words: Stokes flow, variable density, variable viscosity, saddle point problems, projection method, preconditioning, GMRES.

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1 Introduction

Many numerical methods for solving the time-dependent (unsteady) incompressible [1, 3, 24, 27] or low Mach number [14, 43] equations require the solution of a linear unsteady Stokes flow subproblem. The linear steady Stokes problem is of particular interest for low Reynolds number flows [26, 42] or flow in viscous boundary layers. In this work, we investigate efficient linear solvers for the unsteady and steady Stokes equations in the presence of variable density and viscosity. Specifically, we consider the coupled velocity-pressure Stokes system [20, 49]

$$\begin{cases} \rho \boldsymbol{u}_t + \boldsymbol{\nabla} p = \boldsymbol{\nabla}\cdot\boldsymbol{\tau}(\boldsymbol{u}) + \boldsymbol{f}, \\ \boldsymbol{\nabla}\cdot\boldsymbol{u} = g, \end{cases} \tag{1.1}$$

where $\rho(\boldsymbol{r})$ is the density, $\boldsymbol{u}(\boldsymbol{r},t)$ is the velocity, $p(\boldsymbol{r},t)$ is the pressure, $\boldsymbol{f}(\boldsymbol{r},t)$ is a force density, and $\boldsymbol{\tau}(\boldsymbol{u})$ is the viscous stress tensor. A nonzero velocity-divergence $g(\boldsymbol{r},t)$ arises, for example, in low Mach number models because of compositional or temperature variations [43]. The viscous stress $\boldsymbol{\tau}(\boldsymbol{u})$ is $\mu\boldsymbol{\nabla}\boldsymbol{u}$ for constant viscosity incompressible flow, $\mu\left[\boldsymbol{\nabla}\boldsymbol{u}+(\boldsymbol{\nabla}\boldsymbol{u})^T\right]$ when $g=0$ (incompressible flow), and $\mu[\boldsymbol{\nabla}\boldsymbol{u}+(\boldsymbol{\nabla}\boldsymbol{u})^T]+(\gamma-\frac{2}{3}\mu)(\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{I}$ when $g\neq 0$, where $\mu(\boldsymbol{r},t)$ is the shear viscosity and $\gamma(\boldsymbol{r},t)$ is the bulk viscosity. When the inertial term is neglected, $\rho\boldsymbol{u}_t=0$, (1.1) reduces to the time-independent (steady) Stokes equations. In this work we consider periodic boundary conditions and physical boundary conditions that involve velocity only, notably no-slip and free-slip physical boundaries†.

Spatial discretization of (1.1) can be carried out using standard finite-volume or finite-element techniques. Applying the backward Euler scheme to solve the spatially-discretized equations with time step size Δt gives the following discrete system for the velocity $\boldsymbol{u}^{n+1}$ and the pressure p^{n+1} at the end of time step n,

$$\begin{cases} \rho\left(\dfrac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{\Delta t}\right) + \boldsymbol{\nabla} p^{n+1} = \boldsymbol{\nabla}\cdot\boldsymbol{\tau}\left(\boldsymbol{u}^{n+1}\right) + \boldsymbol{f}^{n+1}, \\ \boldsymbol{\nabla}\cdot\boldsymbol{u}^{n+1} = g^{n+1}, \end{cases} \tag{1.2}$$

where $\boldsymbol{f}^{n+1}$ contains external forcing terms such as gravity and any explicitly-handled terms such as, for example, advection. Similar linear systems are obtained with other implicit and semi-implicit temporal discretizations [1, 3, 27]. In the limit $\rho/\Delta t\to 0$, the system (1.2) reduces to the steady Stokes equations. Here we will assume that the spatial discretization is stable, more precisely, that the Stokes system (1.2) is "uniformly solvable" as the spatial discretization becomes finer, i.e., that a suitable measure of the condition number of the Schur complement of (1.2) remains bounded as the grid spacing

†When the normal component of velocity is specified on the whole boundary of the computational domain Ω, a compatibility condition $\int_{\partial\Omega}\boldsymbol{u}\cdot\boldsymbol{n}dS=\int_{\Omega}g d\boldsymbol{r}$ needs to be imposed.