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New Splitting Methods for Convection-Dominated Diffusion Problems and Navier-Stokes Equations

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Abstract. We present a new splitting method for time-dependent convention-dominated diffusion problems. The original convention diffusion system is split into two sub-systems: a pure convection system and a diffusion system. At each time step, a convection problem and a diffusion problem are solved successively. A few important features of the scheme lie in the facts that the convection subproblem is solved explicitly and multistep techniques can be used to essentially enlarge the stability region so that the resulting scheme behaves like an unconditionally stable scheme; while the diffusion subproblem is always self-adjoint and coercive so that they can be solved efficiently using many existing optimal preconditioned iterative solvers. The scheme can be extended for solving the Navier-Stokes equations, where the nonlinearity is resolved by a linear explicit multistep scheme at the convection step, while only a generalized Stokes problem is needed to solve at the diffusion step and the major stiffness matrix stays invariant in the time marching process. Numerical simulations are presented to demonstrate the stability, convergence and performance of the single-step and multistep variants of the new scheme.

AMS subject classifications: 65M12, 65M60, 76D05

Key words: Convention-dominated diffusion problems, Navier-Stokes equations, operator splitting, finite elements, multistep scheme.

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1 Introduction

In this work we shall first propose a new fully discrete splitting scheme for solving the convention-dominated diffusion problems of the following general form

$$u_t+\nabla\cdot(\mathbf{b}u)-\nabla\cdot(\varepsilon\nabla u)+cu=F \quad \text{in} \quad \Omega\times(0,T), \tag{1.1}$$

with the boundary and initial conditions

$$u=u_b \quad \text{on} \quad \partial\Omega\times(0,T); \quad u(0,\mathbf{x})=u_0(\mathbf{x}) \quad \text{in} \quad \Omega, \tag{1.2}$$

where Ω is an open bounded polyhedral domain in $\mathbb{R}^d$ ($d=1,2,3$) with boundary $\Gamma=\partial\Omega$, and $[0,T]$ is the time interval. Functions $\mathbf{b}$ and c in (1.1) are the convective field and reactive coefficient respectively, and $\varepsilon>0$ is a constant diffusion coefficient, while F, u_b and u_0 are the specified source term, the boundary and initial data respectively. As we are mainly interested in the construction of numerical schemes, we will not specify some detailed regularity conditions on all these coefficients to ensure the well-posedness of the initial-boundary value problem (1.1)-(1.2).

Then the new fully discrete splitting scheme will be extended for solving the Navier-Stokes equations

$$\begin{cases} \mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}-Re^{-1}\triangle\mathbf{u}+\nabla p=\mathbf{F} \quad \text{in} \quad \Omega\times(0,T), \\ \nabla\cdot\mathbf{u}=0 \quad \text{in} \quad \Omega\times(0,T), \end{cases} \tag{1.3}$$

with the boundary and initial conditions

$$\mathbf{u}=\mathbf{u}_b \quad \text{on} \quad \partial\Omega\times(0,T); \quad \mathbf{u}(0,\mathbf{x})=\mathbf{u}_0(\mathbf{x}) \quad \text{in} \quad \Omega, \tag{1.4}$$

where $\mathbf{u}$, p, $\mathbf{F}$ and Re are respectively the velocity, the pressure, the body force and the Reynolds number, while $\mathbf{u}_b$ and $\mathbf{u}_0$ are the given boundary and initial data.

The numerical solution of a time-dependent problem requires a discretization in both time and space, and possibly some linearization if the problem is nonlinear. A great variety of time marching schemes are available in the literature, such as the classical methods like the forward and backward Euler schemes, the Crank-Nicolson scheme, the Adams-Bashforth method etc. Operator splitting is also a popular technique for time discretization, such as the Yanenko method, the Peaceman-Rachford method, the Douglas-Rachford method and the θ scheme; see [1–3] and references therein.

In solving the convection-dominated diffusion equations and the Navier-Stokes equations with large Reynolds numbers, it is well known that standard finite element methods perform poorly and may exhibit nonphysical oscillations. Many spatial stabilization techniques have been proposed and studied. The streamline-upwind Petrov-Galerkin method was originally developed in [4, 5] for convective transport problems, and its basic idea is to modify the standard Petrov-Galerkin formulation by adding a streamline upwind perturbation, which acts only in the flow direction and is solely defined in the