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Super-Grid Modeling of the Elastic Wave Equation in Semi-Bounded Domains

N. Anders Petersson* and Björn Sjögreen

Center for Applied Scientific Computing, L-422, LLNL, P.O. Box 808, Livermore, CA 94551, USA.

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Abstract. We develop a super-grid modeling technique for solving the elastic wave equation in semi-bounded two- and three-dimensional spatial domains. In this method, waves are slowed down and dissipated in sponge layers near the far-field boundaries. Mathematically, this is equivalent to a coordinate mapping that transforms a very large physical domain to a significantly smaller computational domain, where the elastic wave equation is solved numerically on a regular grid. To damp out waves that become poorly resolved because of the coordinate mapping, a high order artificial dissipation operator is added in layers near the boundaries of the computational domain. We prove by energy estimates that the super-grid modeling leads to a stable numerical method with decreasing energy, which is valid for heterogeneous material properties and a free surface boundary condition on one side of the domain. Our spatial discretization is based on a fourth order accurate finite difference method, which satisfies the principle of summation by parts. We show that the discrete energy estimate holds also when a centered finite difference stencil is combined with homogeneous Dirichlet conditions at several ghost points outside of the far-field boundaries. Therefore, the coefficients in the finite difference stencils need only be boundary modified near the free surface. This allows for improved computational efficiency and significant simplifications of the implementation of the proposed method in multi-dimensional domains. Numerical experiments in three space dimensions show that the modeling error from truncating the domain can be made very small by choosing a sufficiently wide super-grid damping layer. The numerical accuracy is first evaluated against analytical solutions of Lamb's problem, where fourth order accuracy is observed with a sixth order artificial dissipation. We then use successive grid refinements to study the numerical accuracy in the more complicated motion due to a point moment tensor source in a regularized layered material.

AMS subject classifications: 65M06, 74S20, 86A15

Key words: Far field boundary conditions, higher order methods, elastic wave equation, summation by parts.

*Corresponding author. *Email addresses:* `petersson1@llnl.gov` (N. A. Petersson), `sjogreen2@llnl.gov` (B. Sjogreen)

1 Introduction

To numerically solve a time-dependent wave equation in an unbounded spatial domain, it is necessary to truncate the domain and impose a far-field closure at, or near, the boundaries of the truncated domain. Numerous different approaches have been suggested, see for example [4,6,16]. The perfectly matched layer (PML) technique, originally proposed by Berenger [3] and later improved by many others, has been very successful for electromagnetic wave simulations. Unfortunately, the PML technique sometimes suffers from stability problems when applied to the elastic wave equation, where free surface boundaries and material discontinuities can form wave guides in which the solution of the PML system becomes unstable [18]. The PML system is also known to exhibit stability problems for some anisotropic wave equations [2].

Similar to the PML technique, the super-grid method [1] modifies the original wave equation in layers near the boundary of the computational domain. The PML system is defined by Fourier transforming the original wave equation in time and applying a frequency-dependent complex-valued coordinate transformation in the layers. Additional dependent variables, governed by additional differential equations, must be introduced to define the PML system in the time domain. In comparison, the super-grid method is based on applying a real-valued coordinate stretching in the layers, where also artificial dissipation is added. The super-grid method does not rely on additional dependent variables, and is therefore more straight forward to implement. In the layers near the boundary, the PML method damps the waves; in contrast, the super-grid method both damps the waves and slows them down. The main advantage over the PML technique is that the solution of the wave equation with super-grid layers is energy stable, if there is a corresponding energy estimate for the underlying wave equation.

In this article, we generalize the super-grid approach [1] to the elastic wave equation in second order formulation. Motivated by applications from seismology and seismic exploration, we focus on half-plane or half-space domains, where a free surface boundary condition must be satisfied on only one side of the domain. The half-space problem subject to a free surface condition permits surface waves. These waves only propagate along the free surface and decay exponentially away from the surface. They are fundamentally different from the longitudinal and transverse waves that travel through the volume of the domain. Surface waves therefore constitute an additional type of wave that need to be absorbed by the far-field closure.

We are primarily interested in cases where the solution is of a transient nature, being driven by initial data with compact support, or by a forcing function that only is active (non-zero) for a limited time. Because of the artificial damping in the super-grid layers, the solution becomes very small on the outside of the layers. For this reason, it is natural to impose homogeneous Dirichlet conditions at the super-grid boundaries, which truncate the computational domain. In this paper, we develop a finite difference method where fourth order accurate summation by parts (SBP) operators [17] are combined with centered fourth order accurate finite difference formulas in the interior of the