Commun. Comput. Phys.
doi: 10.4208/cicp.111213.110414a

Vol. **16**, No. 4, pp. 1102-1115
October 2014

Dirichlet-to-Neumann Mapping for the Characteristic Elliptic Equations with Symmetric Periodic Coefficients

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Received 11 December 2013; Accepted (in revised version) 11 April 2014

Available online 12 August 2014

Abstract. Based on the numerical evidences, an analytical expression of the Dirichlet-to-Neumann mapping in the form of infinite product was first conjectured for the one-dimensional characteristic Schrödinger equation with a sinusoidal potential in [Commun. Comput. Phys., 3(3): 641-658, 2008]. It was later extended for the general second-order characteristic elliptic equations with symmetric periodic coefficients in [J. Comp. Phys., 227: 6877-6894, 2008]. In this paper, we present a proof for this Dirichlet-to-Neumann mapping.

AMS subject classifications: 65M99, 81-08

Key words: Dirichlet-to-Neumann mapping, Schrödinger equation, symmetric periodic potentials, absorbing boundary conditions.

1 Introduction

Periodic structure problems largely exist in the science and engineering such as semiconductor nanostructures, semiconductor superlattices [3,24], photonic crystals (PC) structures [2,15,18], meta materials [20] and Bragg gratings of surface plasmon polariton (SPP) waveguides [11,19]. Usually they are modeled by partial differential equations with periodic coefficients and/or periodic geometries. In order to numerically solve these equations efficiently, one usually confines the computational domain by introducing artificial boundaries and imposing suitable boundary conditions on them. For wave-like equations, the ideal boundary conditions should not only lead to well–posed problems, but also mimic the perfect absorption of waves which travel out of the computational domain

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through the artificial boundaries. Right in this sense, these boundary conditions are usually called absorbing (or transparent, non-reflecting in the same spirit) in the literature.

The study of absorbing boundary conditions (ABCs) for linear wave-like equations has been a hot research topic for many years and significant developments have been made on their designing and implementing. The interested reader is referred to the review papers [1, 8, 10, 23]. Comparatively, the study of exact or approximate ABCs for periodic structure problems is relatively a very current research topic, cf. the recent papers [6,7,14,21,22,25,26]. For a review on the theory and numerical techniques of waves in locally periodic media we refer the reader to [4,9,12].

Based on the numerical evidences, one of the authors [28] conjectured an exact ABC in the form of Dirichlet-to-Neumann (DtN) mapping for the characteristic Schrödinger equation

$$\left[-\partial_x^2+V(x)\right]y=zy, \tag{1.1}$$

when the potential V is sinusoidal. This DtN mapping is expressed as an infinite product

$$\frac{y'(0)}{y(0)}=-\sqrt{-z+\mu_{0,V}^{NN}}\prod_{m=1}^{+\infty}\frac{\sqrt{-z+\mu_{m,V}^{NN}}}{\sqrt{-z+\mu_{m,V}^{DD}}}, \quad \Im z>0, \tag{1.2}$$

where $\{\mu_{r,V}^{NN}\}_{r\geq 0}$ are eigenvalues of the Schrödinger operator $-\partial_x^2+V(x)$ with Neumann boundary conditions imposed on the periodic cell boundaries, and $\{\mu_{r,V}^{DD}\}_{r\geq 1}$ are eigenvalues with Dirichlet boundary conditions imposed. This DtN mapping was then extended in [5] for the general second-order characteristic elliptic equations

$$\left[-\partial_x m^{-1}(x)\partial_x+V(x)\right]y=\rho(x)zy, \quad \forall x\geq 0, \tag{1.3}$$

where ρ, V and m are supposed to be symmetric periodic. The exact DtN mapping for (1.3) was conjectured to be

$$\frac{y'(0)}{y(0)}=-\sqrt{c(0)\rho(0)}\sqrt{-z+\mu_{0,V}^{NN}}\prod_{m=1}^{+\infty}\frac{\sqrt{-z+\mu_{m,V}^{NN}}}{\sqrt{-z+\mu_{m,V}^{DD}}}, \quad \Im z>0, \tag{1.4}$$

where again, $\{\mu_{r,V}^{NN}\}_{r\geq 0}$ are the eigenvalues of characteristic elliptic equations (1.3) with Neumann boundary conditions and $\{\mu_{r,V}^{DD}\}_{r\geq 1}$ with Dirichlet boundary conditions specified on the periodic cell boundaries. The exact DtN mapping (1.4) was used in [5] to design the exact ABCs for a set of one-dimensional time-dependent wave equations with symmetric periodic coefficients.

In this paper, we give a proof for the DtN mapping expression (1.4). Our idea was stimulated by the work of J. Pöschel and E. Trubowitz. In [17], they considered the characteristic Schrödinger equation (1.1) with 1-periodic potentials. For the basic solution φ