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Approximation of Spatio-Temporal Random Processes Using Tensor Decomposition

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Abstract. A new representation of spatio-temporal random processes is proposed in this work. In practical applications, such processes are used to model velocity fields, temperature distributions, response of vibrating systems, to name a few. Finding an efficient representation for any random process leads to encapsulation of information which makes it more convenient for a practical implementations, for instance, in a computational mechanics problem. For a single-parameter process such as spatial or temporal process, the eigenvalue decomposition of the covariance matrix leads to the well-known Karhunen-Loève (KL) decomposition. However, for multiparameter processes such as a spatio-temporal process, the covariance function itself can be defined in multiple ways. Here the process is assumed to be measured at a finite set of spatial locations and a finite number of time instants. Then the spatial covariance matrix at different time instants are considered to define the covariance of the process. This set of square, symmetric, positive semi-definite matrices is then represented as a third-order tensor. A suitable decomposition of this tensor can identify the dominant components of the process, and these components are then used to define a closed-form representation of the process. The procedure is analogous to the KL decomposition for a single-parameter process, however, the decompositions and interpretations vary significantly. The tensor decompositions are successfully applied on (i) a heat conduction problem, (ii) a vibration problem, and (iii) a covariance function taken from the literature that was fitted to model a measured wind velocity data. It is observed that the proposed representation provides an efficient approximation to some processes. Furthermore, a comparison with KL decomposition showed that the proposed method is computationally cheaper than the KL, both in terms of computer memory and execution time.

AMS subject classifications: 60G12, 65F99, 65D15

Key words: Random process, spatio-temporal process, tensor decomposition, uncertainty quantification, probabilistic mechanics.

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1 Introduction

In a probabilistic treatment of uncertainties in analyzing and designing physical systems, random processes are used to describe and model various parameters and phenomena. Sources of these uncertainties can be attributed to insufficient data, variability in manufacturing process, error incurred during mathematical idealization of the problem, to mention a few. Let $(\Omega,\mathcal{F},\mu)$ denote a probability space where Ω denotes the set of elementary events θ, $\mathcal{F}$ denotes a σ-algebra on this event set, and μ denotes the probability measure. Let $\boldsymbol{x}\in\mathbb{R}^d$ denote a spatial location where $d=1,2$ or 3, and $t\in\mathbb{R}^+$ denote the time. Then the heterogeneity of Young's modulus of a solid can be modeled as a spatial random process or field $u(\boldsymbol{x},\theta)$, a time-varying excitation can be modeled as a temporal random field $u(t,\theta)$. Similarly the parameters that are dependent upon both space and time — such as a velocity field of a fluid in motion, temperature field, dynamic response of a large structure — can be modeled as spatio-temporal process $u(\boldsymbol{x},t,\theta)$. In this work a spatially and temporally discrete version of the real-valued processes is considered, that is, the processes are measured or evaluated at spatial locations $\boldsymbol{x}_i:\ i=1,2,\cdots,N_s$ and time instants $t_j:\ j=1,2,\cdots,N_t$. Therefore, the spatio-temporal processes can now be written in the following matrix form

$$\boldsymbol{U}(\theta)=\left[\begin{array}{cccc} u(\boldsymbol{x}_1,t_1,\theta) & u(\boldsymbol{x}_1,t_2,\theta) & \cdots & u(\boldsymbol{x}_1,t_{N_t},\theta) \\ u(\boldsymbol{x}_2,t_1,\theta) & u(\boldsymbol{x}_2,t_2,\theta) & \cdots & u(\boldsymbol{x}_2,t_{N_t},\theta) \\ \vdots & \vdots & \vdots & \vdots \\ u(\boldsymbol{x}_{N_s},t_1,\theta) & u(\boldsymbol{x}_{N_s},t_2,\theta) & \cdots & u(\boldsymbol{x}_{N_s},t_{N_t},\theta) \end{array}\right]\in\mathbb{R}^{(N_s\times N_t)}. \tag{1.1}$$

The spatial and temporal processes can accordingly be expressed in a vector form. However, this explicit form, which is often needed for computational purpose, is not known in most practical cases. Either a few realizations of the process or some information about the covariance is known. Therefore a representation of the process needs to be found using this available information. In the current work, it is assumed that the only available information are the mean and spatial covariance for a set of time instants.

Let the mathematical expectation operator $\int_\Omega \cdot d\mu(\theta)$ be denoted as $\mathbb{E}\{\cdot\}$. Then, for a single-parameter process such as a spatial process $u(\boldsymbol{x},\theta)$ the covariance between two spatial locations $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is defined as

$$\begin{aligned} & Cov(u(\boldsymbol{x}_1,\theta),u(\boldsymbol{x}_2,\theta)) \\ =& \mathbb{E}\{(u(\boldsymbol{x}_1,\theta)-\bar{u}(\boldsymbol{x}_1))(u(\boldsymbol{x}_2,\theta)-\bar{u}(\boldsymbol{x}_2))\}, \end{aligned} \tag{1.2}$$

with $\bar{u}\in\mathbb{R}$ denoting the mean of the process. A few largest eigenvalues and corresponding eigenvectors of this $(N_s\times N_s)$ symmetric positive-semidefinite covariance matrix hold a significant amount of information about the process $u(\boldsymbol{x},\theta)$. These eigenvectors serve as the set of bases in an approximate representation of this process, known as the Karhunen-Loève (KL) decomposition [1–5]. Similarly, the covariance matrix for a spatio-temporal process can be constructed with the elements as $Cov(u(\boldsymbol{x}_i,t_k,\theta),u(\boldsymbol{x}_j,t_l,\theta)),\ i,j=1,2,\cdots,N_s$