Commun. Comput. Phys.
doi: 10.4208/cicp.240113.071113a

Vol. **16**, No. 1, pp. 1-34
July 2014

Adaptive Bayesian Inference for Discontinuous Inverse Problems, Application to Hyperbolic Conservation Laws

Alexandre Birolleau[1,2], Gaël Poëtte[1] and Didier Lucor[2,3,*]

[1] *CEA, DAM, DIF, F-91297 Arpajon, France.*
[2] *UPMC Univ Paris 06, UMR 7190, Institut Jean Le Rond d'Alembert, F-75005 Paris, France.*
[3] *CNRS, UMR 7190, Institut Jean Le Rond d'Alembert, F-75005 Paris, France.*

Received 24 January 2013; Accepted (in revised version) 7 November 2013

Available online 28 March 2014

Abstract. Various works from the literature aimed at accelerating Bayesian inference in inverse problems. Stochastic spectral methods have been recently proposed as surrogate approximations of the forward uncertainty propagation model over the support of the prior distribution. These representations are efficient because they allow affordable simulation of a large number of samples from the posterior distribution. Unfortunately, they do not perform well when the forward model exhibits strong nonlinear behavior with respect to its input.

In this work, we first relate the fast (exponential) L^2-convergence of the forward approximation to the fast (exponential) convergence (in terms of Kullback-Leibler divergence) of the approximate posterior. In particular, we prove that in case the prior distribution is *uniform*, the posterior is at least twice as fast as the convergence rate of the forward model in those norms. The Bayesian inference strategy is developed in the framework of a stochastic spectral projection method. The predicted convergence rates are then demonstrated for simple nonlinear inverse problems of varying smoothness.

We then propose an efficient numerical approach for the Bayesian solution of inverse problems presenting strongly nonlinear or discontinuous system responses. This comes with the improvement of the forward model that is *adaptively* approximated by an iterative generalized Polynomial Chaos-based representation. The numerical approximations and predicted convergence rates of the former approach are compared to the new iterative numerical method for nonlinear time-dependent test cases of varying dimension and complexity, which are relevant regarding our hydrodynamics motivations and therefore regarding hyperbolic conservation laws and the apparition of discontinuities in finite time.

AMS subject classifications: 52B10, 65D18, 68U05, 68U07

Key words: Stochastic inverse problems, Bayesian inference, iterative generalized Polynomial Chaos, compressible gas dynamics.

*Corresponding author. *Email addresses:* `alexandre.birolleau@cea.fr` (A. Birolleau), `gael.poette@cea.fr` (G. Poëtte), `didier.lucor@upmc.fr` (D. Lucor)

1 Introduction

Nowadays, the development of efficient computational tools to support decision-making and risk analysis under *uncertainty* is critical for the design and operation of engineered systems and more generally for *reliable* predictive science. An open question with a huge significance for *uncertainty quantification* (UQ) is the problem of realistic representation of *input* uncertainty (initial/operating/boundary conditions, model parameters, source terms, $\cdots$) to the model. A quick survey of the UQ literature shows that research in this area has been accustomed to the development of the propagation step and quantification of the response, with improvement on the efficiency, implementation, performance, $\cdots$. In many works, the quantification of input uncertainty is often rudimentary, associating a random variable to each of the random parameter, and often making *a priori* choice on the distributions, relying for instance on *labelled* distributions, such as *uniform* distributions, due to a lack of knowledge. Another weakness is the assumption of random parameters *independence*. Indeed, the *gold rush* for the development of suitable and efficient stochastic representations of ever increasing larger data sets (e.g., hundreds of random parameters) relies heavily on the assumption of independent random dimensions which is most of the time not justified for engineering systems. In fact the effective stochastic dimensionality of the system depends strongly on the appropriate representation of the correlations existing between the dependent random variables representing the inputs. This mathematical description is particularly difficult when data is gathered from different sources, let say from both experiments and simulations, or when direct observations/measurements are not possible or too costly. Several methodologies for the identification of representations of random variables/processes from *experimental* data for instance have been proposed, such as the method of moments [2], maximum likelihood [8, 17], maximum entropy [7] or Bayesian inference [16, 77]. Inverse problems (IP) usually refer to the estimation of model parameters or inputs from *indirect* observations. While the resolution of a forward model predicts the system outputs given the inputs by solving the governing equations, the IP reverses this relationship by seeking to estimate uncertain inputs from measurements or observations. The IP is often formulated as a (large) *deterministic nonlinear* optimization problem that minimizes the discrepancy between the observed and predicted outputs in some appropriate norm while also minimizing a regularization term that penalizes unwanted features of the inputs [27, 65]. Following this procedure, a set of *best* inputs, i.e., fitting the data and minimizing the regularization penalty term, are obtained. Nevertheless, the predictive accuracy strongly depends on the availability of large input data sets. In practice the observations are *limited* and often *noisy*. Therefore, it becomes more legitimate to seek a complete *statistical* description of the input values that is consistent with the data, instead of discrete estimates of the best-fit inputs.

The Bayesian inference follows this path by reformulating the IP as a problem of *statistical inference*, incorporating the forward model, prior information on the inputs, and uncertainties in the measurements. The solution is the *posterior* joint pdf of the inputs,