Commun. Comput. Phys.
doi: 10.4208/cicp.251112.211013a

Vol. **15**, No. 5, pp. 1461-1479
May 2014

A Numerical Study on the Weak Galerkin Method for the Helmholtz Equation

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Received 25 November 2012; Accepted (in revised version) 21 October 2013

Communicated by Jan S. Hesthaven

Available online 12 March 2014

Abstract. A weak Galerkin (WG) method is introduced and numerically tested for the Helmholtz equation. This method is flexible by using discontinuous piecewise polynomials and retains the mass conservation property. At the same time, the WG finite element formulation is symmetric and parameter free. Several test scenarios are designed for a numerical investigation on the accuracy, convergence, and robustness of the WG method in both inhomogeneous and homogeneous media over convex and non-convex domains. Challenging problems with high wave numbers are also examined. Our numerical experiments indicate that the weak Galerkin is a finite element technique that is easy to implement, and provides very accurate and robust numerical solutions for the Helmholtz problem with high wave numbers.

AMS subject classifications: 65N15, 65N30, 76D07, 35B45, 35J50

Key words: Galerkin finite element methods, discrete gradient, Helmholtz equation, large wave numbers, weak Galerkin.

1 Introduction

We consider the Helmholtz equation of the form

$$-\nabla\cdot(d\nabla u)-k^2u=f \qquad \text{in } \Omega, \tag{1.1a}$$

$$\nabla u\cdot\mathbf{n}-iku=g \qquad \text{on } \partial\Omega, \tag{1.1b}$$

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where $k>0$ is the wave number, $f\in L^2(\Omega)$ represents a harmonic source, $g\in L^2(\partial\Omega)$ is a given data function, and $d=d(x,y)>0$ is a spatial function describing the dielectric properties of the medium. Here Ω is a polygonal or polyhedral domain in $\mathbb{R}^d$ $(d=2,3)$.

Under the assumption that the time-harmonic behavior is assumed, the Helmholtz equation (1.1a) governs many macroscopic wave phenomena in the frequency domain including wave propagation, guiding, radiation and scattering. The numerical solution to the Helmholtz equation plays a vital role in a wide range of applications in electromagnetics, optics, and acoustics, such as antenna analysis and synthesis, radar cross section calculation, simulation of ground or surface penetrating radar, design of optoelectronic devices, acoustic noise control, and seismic wave propagation. However, it remains a challenge to design robust and efficient numerical algorithms for the Helmholtz equation, especially when large wave numbers or highly oscillatory solutions are involved [37].

For the Helmholtz problem (1.1a)-(1.1b), the corresponding variational form is given by seeking $u\in H^1(\Omega)$ satisfying

$$(d\nabla u,\nabla v)-k^2(u,v)+ik\langle u,v\rangle_{\partial\Omega}=(f,v)+\langle g,v\rangle_{\partial\Omega},\quad \forall v\in H^1(\Omega), \tag{1.2}$$

where $(v,w)=\int_\Omega vwdx$ and $\langle v,w\rangle_{\partial\Omega}=\int_{\partial\Omega}vwds$. In a classic finite element procedure, continuous polynomials are used to approximate the true solution u. In many situations, the use of discontinuous functions in the finite element approximation often provides the methods with much needed flexibility to handle more complicated practical problems. However, for discontinuous polynomials, the strong gradient ∇ in (1.2) is no longer meaningful. Recently developed weak Galerkin finite element methods [33,38,39] provide means to solve this difficulty by replacing the differential operators by the weak forms as distributions for discontinuous approximating functions.

Weak Galerkin (WG) methods refer to general finite element techniques for partial differential equations and were first introduced and analyzed in [33] for second order elliptic equations. Through rigorous error analysis, optimal order of convergence of the WG solution in both discrete H^1 norm and L^2 norm is established under minimum regularity assumptions in [33]. The mixed weak Galerkin finite element method is studied in [34]. The WG methods are by design using discontinuous approximating functions.

In this paper, we will apply WG finite element methods [33,38,39] to the Helmholtz equation. The WG finite element approximation to (1.2) can be derived naturally by simply replacing the differential operator gradient ∇ in (1.2) by a weak gradient ∇_w: find $u_h\in V_h$ such that for all $v_h\in V_h$ we have

$$(d\nabla_w u_h,\nabla_w v_h)-k^2(u_0,v_0)+ik\langle u_b,v_b\rangle_{\partial\Omega}=(f,v_0)+\langle g,v_b\rangle_{\partial\Omega}, \tag{1.3}$$

where u_0 and u_b represent the values of u_h in the interior and the boundary of the triangle respectively. The weak gradient ∇_w will be defined precisely in the next section. We note that the weak Galerkin finite element formulation (1.3) is simple, symmetric and parameter free.

To fully explore the potential of the WG finite element formulation (1.3), we will investigate its performance for solving the Helmholtz problems with large wave numbers.