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Numerical Analysis of an Adaptive FEM for Distributed Flux Reconstruction

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Abstract. This paper studies convergence analysis of an adaptive finite element algorithm for numerical estimation of some unknown distributed flux in a stationary heat conduction system, namely recovering the unknown Neumann data on interior inaccessible boundary using Dirichlet measurement data on outer accessible boundary. Besides global upper and lower bounds established in [23], a posteriori local upper bounds and quasi-orthogonality results concerning the discretization errors of the state and adjoint variables are derived. Convergence and quasi-optimality of the proposed adaptive algorithm are rigorously proved. Numerical results are presented to illustrate the quasi-optimality of the proposed adaptive method.

AMS subject classifications: 65N21, 65N50, 65N30

Key words: Inverse problems, adaptive finite element method, a posteriori error estimates, quasi-orthogonality, convergence analysis.

1 Introduction

In this paper we study convergence analysis of an adaptive finite element algorithm for numerical estimation of some unknown distributed flux in a stationary heat conduction system, namely recovering the unknown Neumann data, called *fluxes* in the sequel, on the interior inaccessible boundary using Dirichlet measurement data on the outer accessible boundary.

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In heat conduction, the flux distribution is of paramount practical interest, e.g., the real-time monitoring in steel industry [1], the visualization by liquid crystal thermography [18], and estimating the freezing front velocity in the solidification process [37]. But its accurate distribution is rather difficult to obtain on some inaccessible boundary, such as the interior boundary of nuclear reactors and steel furnaces. Engineers seek to estimate them from accessible outer boundary measurements, which naturally gives rise to the inverse problem of estimating the distribution of fluxes. This inverse problem is essentially lack of continuous dependance on data, thus ill-posed in Hadamard's sense [22].

Numerous numerical investigations have been made for this distributed flux reconstruction problem, among which the least-squares formulation [36–38] has received intensive investigations and it has been implemented by means of the boundary integral method [38] and the finite element method [36]. Recently, adaptive techniques are introduced for this problem for efficiency consideration [23]. Guided by the a posteriori error estimates, the adaptive algorithm in [23] automatically refines the mesh to better approximate the local but potentially very important features of the distributed flux, e.g., non-smooth boundaries, discontinuous fluxes, or singular fluxes with spikes or abrupt sign changes. The computational cost of the adaptive algorithm is significantly reduced, compared with that of the uniform refinement.

The research on Adaptive Finite Element Methods (AFEMs) dates back to the seminal work [2] in the late 1970s. The main themes in AFEMs is how to measure, control and effectively minimize the discretization error of quantities of interest based on the computed solution and given data, among which the difficulties lies in 1) deriving a posteriori error estimates and 2) proving convergence of the resulting adaptive algorithm based on those a posteriori error estimates.

AFEMs have witnessed significant advance in reducing the computational complexity and improving efficiency in the solution of a variety of partial differential equations in the past decades. In inverse problems and control community, adaptive methods have been applied with emphasis on deriving a posteriori error estimates, to mention a few: 1) the dual weighted residual framework in terms of some quantity of interest [3,4,9], which provides a general recipe to construct a posteriori error estimates; 2) adaptive parameter identification in elliptic systems [20]; 3) adaptive methods in PDE-constrained optimal control problems [25–27].

On the other hand, theoretical convergence analysis of adaptive finite element methods has made important progress in recent years. It started with Dörfler [19], who introduced a crucial marking strategy, from now on called Dörfler's marking strategy, and proved the strict energy reduction property for the Laplace equation provided the initial mesh satisfies the fineness assumption. Some breakthrough has been made on the optimality of adaptive finite element methods since the pioneering work of Binev, Dahmen, DeVore [10] and Stevenson [34]. In fact, the question of convergence and optimality of adaptive finite element methods has been subject to intensive studies in recent years, which is reflected from vast literature, see, e.g., [5–7, 12–15, 28, 30–32]. However, most references aforementioned focused on direct model problems, as opposed to few litera-