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A Unstructured Nodal Spectral-Element Method for the Navier-Stokes Equations

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Abstract. An unstructured nodal spectral-element method for the Navier-Stokes equations is developed in this paper. The method is based on a triangular and tetrahedral rational approximation and an easy-to-implement nodal basis which fully enjoys the tensorial product property. It allows arbitrary triangular and tetrahedral mesh, affording greater flexibility in handling complex domains while maintaining all essential features of the usual spectral-element method. The details of the implementation and some numerical examples are provided to validate the efficiency and flexibility of the proposed method.

AMS subject classifications: 65N35, 65N22, 65F05, 35J05

Key words: Navier-Stokes equations, unstructured mesh, triangular spectral element method.

1 Introduction

The spectral-element method combines the geometric flexibility of finite elements with the high accuracy of spectral methods. It exhibits several favorable computational properties, such as the use of tensor products, naturally diagonal mass matrices, and suitability for parallel computation.

However, in order to use the properties of the tensor product, the standard spectral-element method is usually limited to quadrilateral/hexahedral partitions. This requirement makes it difficult to use unstructured mesh for complex geometries. One way to overcome this drawback is to allow the use of triangles/tetrahedrons in the partition. There has been a number of works addressing the so-called triangular spectral methods. The existing spectral methods on triangle can be classified into different types according

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to the class of functions used in the approximations: (i) approximations by polynomials in triangle through mapping (see, e.g., [2,5,9,14,19]); (ii) approximations by polynomials in triangle using special nodal points such as Fekete points (see, e.g., [8,15,16,20]); and (iii) approximations by non-polynomial functions in triangle (see, e.g., [1,7,18]).

The triangular spectral method based on polynomial spaces were motivated by the classical result that any smooth function can be well approximated by polynomials. Koornwinder [10] and Dubiner [5] constructed the orthogonal polynomials on triangle, often referred as the Dubiner's basis in the spectral-element community. The first practical implementation of the Dubiner basis in the solution of incompressible Navier-Stokes equations was carried out in the spectral-element package *NekTar* [9,19]. Their approach is based on the *modal* formulation in which the basis functions are Jacobi polynomials with index varying with the polynomial order. A drawback of this modal basis is that there is no corresponding *nodal* basis, making it more difficult, to implement. Recently, a triangular spectral method using rational polynomials was proposed and analyzed for elliptic problems (cf. [18]). It is extended to the Stokes problem on a triangle in [3]. The main advantage of this method is that a *nodal* basis is available, so it can be incorporated into the usual nodal spectral-element framework. In particular, it preserves the tensor product structure which enables the fast evaluation of the matrix-vector multiplications.

In this paper, we design an unstructured spectral-element method for the Navier-Stokes equations. More precisely, we describe in detail how to formulate the nodal basis with two-dimensional unstructured meshes for the Stokes equations, and apply them for solving time dependent Navier-Stokes equations. Our nodal basis functions in triangles/tetrahedrons are constructed from the standard tensor product of Lagrangian polynomials defined on the 2-D Gauss-Lobatto points through the Duffy mapping. The advantages of this nodal basis are that it allows arbitrary mixture of triangular and rectangular elements; enjoys the fully tensorial-product property, and can be easily incorporated into an existing spectral-element code.

The paper is organized as follows. In the next section, we first present the rational spectral-element method for the Stokes equations on a single triangle. In Section 3, we extend the method to 2-D unstructured mesh with arbitrary mixture of triangular and quadrilateral elements. In Section 4, we apply this spectral-element method to the Navier-Stokes equations and present several numerical experiments exhibiting its flexibility and accuracy. Some concluding remarks are given in the final section.

2 Preliminaries

In this section, we introduce some necessary notations and recall briefly the triangular spectral method for the Stokes equations developed recently in [3].

Throughout this paper, we use boldface letters to denote vectors and vector functions. Let c be a generic positive constant independent of any functions and of any discretization parameters. We use the expression $A \lesssim B$ to mean that $A \leqslant cB$, and use the expression