Commun. Comput. Phys.
doi: 10.4208/cicp.300410.300710a

Vol. **10**, No. 2, pp. 474-508
August 2011

Local Discontinuous Galerkin Methods for the Degasperis-Procesi Equation

Yan Xu[1,*] and Chi-Wang Shu[2]

[1] *Department of Mathematics, University of Science and Technology of China, Hefei 230026, Anhui, China.*
[2] *Division of Applied Mathematics, Brown University, Providence, RI 02912, USA.*

Received 30 April 2010; Accepted (in revised version) 30 July 2010

Available online 28 April 2011

Abstract. In this paper, we develop, analyze and test local discontinuous Galerkin (LDG) methods for solving the Degasperis-Procesi equation which contains nonlinear high order derivatives, and possibly discontinuous or sharp transition solutions. The LDG method has the flexibility for arbitrary h and p adaptivity. We prove the L^2 stability for general solutions. The proof of the total variation stability of the schemes for the piecewise constant P^0 case is also given. The numerical simulation results for different types of solutions of the nonlinear Degasperis-Procesi equation are provided to illustrate the accuracy and capability of the LDG method.

AMS subject classifications: 65M60, 35Q53

Key words: Local discontinuous Galerkin method, Degasperis-Procesi equation, L^2 stability, total variation stability.

1 Introduction

In this paper, we consider numerical approximations to the Degasperis-Procesi (DP) equation

$$u_t - u_{txx} + 4f(u)_x = f(u)_{xxx}, \tag{1.1}$$

where $f(u)=u^2/2$. We develop two local discontinuous Galerkin (LDG) methods for this nonlinear DP equation. Our proposed schemes are high order accurate, nonlinear stable and flexible for arbitrary h and p adaptivity. The proof of the L^2 stability of the schemes are given for general solutions and total variation stability for the piecewise constant P^0 case is also given. To our best knowledge, this is the first provably stable finite element method for the DP equation.

*Corresponding author. *Email addresses:* `yxu@ustc.edu.cn` (Y. Xu), `shu@dam.brown.edu` (C.-W. Shu)

Degasperis and Procesi [14] studied the following family of third order dispersive PDE conservation laws,

$$u_t + c_0 u_x + \kappa u_{xxx} - \epsilon^2 u_{txx} = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x, \tag{1.2}$$

where κ, ϵ, c_0, c_1, c_2, and c_3 are real constants. Their motivation was to answer the question of which equations of a form similar to the Camassa-Holm (CH) equation are integrable. Applying the method of asymptotic integrability to the family (1.2), they found that there are only three equations that satisfy the asymptotic integrability condition within this family, namely, the KdV equation ($\epsilon = c_2 = c_3 = 0$), the CH equation ($c_1 = -3c_3/2\epsilon^2$, $c_2 = c_3/2$) and one new equation ($c_1 = -2c_3/2\epsilon^2$, $c_2 = c_3$, the DP equation). By rescaling, shifting the dependent variable and applying a Galilean boost [13], one can find the Degasperis-Procesi equation (1.1) which has a similar form to the limiting case of the Camassa-Holm shallow water equation.

Despite the similarities to the CH equation, we would like to point out that the DP equation is truly different. One of the important features of the DP equation is that it has not only peaked solutions [13], for example, $u(x,t) = ce^{-|x-ct|}$, but also shock waves to the equation [9,24], for example

$$u(x,t) = -\frac{1}{t+c}\operatorname{sign}(x)e^{-|x|}, \quad c > 0. \tag{1.3}$$

Also, these two equations have entirely different forms of conservation laws:

- Three useful conservation laws for the DP equation:

$$E_1(u) = \int_R (u - u_{xx})dx, \quad E_2(u) = \int_R (u - u_{xx})v dx, \quad E_3(u) = \int_R u^3 dx,$$

  where $4v - v_{xx} = u$.

- Three useful conservation laws for the CH equation:

$$H_1(u) = \int_R (u - u_{xx})dx, \quad H_2(u) = \int_R (u^2 + u_x^2)dx, \quad H_3(u) = \int_R (u^3 + u u_x^2)dx.$$

We can see that the corresponding conservation laws of the DP equation are much weaker than those of the CH equation. The conservation laws $E_i(u)$ can not guarantee the boundedness of the slope of a wave in the L^2-norm. There is no way to find conservation laws controlling the H^1-norm, which plays a very important role in studying the CH equation. Such nonlinearly dispersive partial differential equations support peakon solutions and shock solutions. The lack of smoothness of the solution introduces more difficulty in the numerical computation. It is a challenge to design stable and high order accurate numerical schemes for solving this equation.

In the last ten years, a lot of analysis has been given for the DP equation. Coclite and Karlsen proved existence and uniqueness results for entropy weak solutions belonging to the class $L^1 \cap BV$ in [9] and uniqueness result for entropy weak solutions by replacing the Kružkov-type entropy inequalities by an Oleinik-type estimate in [10]. For