Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2018-0307

Vol. **27**, No. 3, pp. 672-724
March 2020

An Explicit MUSCL Scheme on Staggered Grids with Kinetic-Like Fluxes for the Barotropic and Full Euler System

Thierry Goudon[1], Julie Llobell[2] and Sebastian Minjeaud[2,*]

[1] *Université Côte d'Azur, Inria, CNRS, LJAD, France.*
[2] *Université Côte d'Azur, CNRS, Inria, LJAD, France.*

Communicated by Kun Xu

Received 27 November 2018; Accepted (in revised version) 11 July 2019

Abstract. We present a second order scheme for the barotropic and full Euler equations. The scheme works on staggered grids, with numerical unknowns stored at dual locations, while the numerical fluxes are derived in the spirit of kinetic schemes. We identify stability conditions ensuring the positivity of the discrete density and energy. We illustrate the ability of the scheme to capture the structure of complex flows with 1D and 2D simulations on MAC grids.

AMS subject classifications: 65M08, 76M12, 76NXX, 35Q31

Key words: Euler equations, finite volume methods, staggered grids, MUSCL schemes.

1 Introduction

This work is concerned with the development of numerical schemes on staggered grids for the Euler equations. Using staggered grids is non standard for the discretization of hyperbolic system (see e.g. [11,19,26,36,50]), since, when stored on a collocated grid, the unknowns of the system are usually gathered in a single vector-valued unknown allowing to identify the wave structure of the system in order to built upwinding techniques. The motivation of the use of staggered grids comes from the attempt to have an unified approach with an incompressible code, see e.g. [25,28,51,53–55]. This is particularly relevant when dealing with low-Mach simulations since letting the Mach number go to 0 enforces incompressibility and collocated approaches might lead to numerical difficulties in this regime, and to the development of spurious instabilities due to an "odd-even

*Corresponding author. *Email addresses:* thierry.goudon@inria.fr (T. Goudon), julie.llobell@laposte.net (J. Llobell), Sebastian.Minjeaud@unice.fr (S. Minjeaud)

decoupling", see [20, 21, 57] and the references therein. This is also of interest in multi-fluid flows simulations that involve additional solenoidal constraints on a velocity field, see e.g. [6, 16, 17, 47]: coupled with a projection approach, the staggered method makes the discretization of the mass conservation equations for all the species interacting in the mixture and the definition of the pressure field (the Lagrange multiplier associated to the solenoidal constraint) compatible.

We first deal with the barotropic Euler system

$$\begin{cases} \partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \boldsymbol{\nabla} \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \boldsymbol{\nabla} (p(\rho)) = 0. \end{cases} \tag{1.1}$$

This model describes the evolution of a compressible fluid (in the absence of external forces). The unknowns ρ and $\mathbf{u}$ stand respectively for the local density and velocity field of the fluid. They depend on the time and space variables, $t \geqslant 0$ and $x \in \mathbb{R}^N$. The model assumes that the pressure p depends on the density ρ only. Here and below, we suppose that the pressure law $\rho \mapsto p(\rho)$ belongs to $\mathcal{C}^2([0,\infty))$ and satisfies

$$p(\rho) > 0, \quad p'(\rho) > 0, \quad p''(\rho) \geqslant 0, \qquad \forall \rho > 0.$$

For instance, these properties hold for the classical power-law $p(\rho) = \lambda \rho^\gamma$ with $\lambda > 0$ and $\gamma > 1$. We refer the reader to the classical treatises [11, 19, 26, 36, 50] for a thorough introduction to these equations and for a description of the numerical issues. Our aim is here to extend at the second order and to higher dimension the scheme introduced in [5]. This scheme is characterized by the following two main features:

- first of all, as said previously, it works on staggered grids, meaning that densities and velocities are stored on different grid points;
- second of all, the fluxes are defined with a flavor of kinetic schemes [18, 22, 23, 32, 44, 45].

Consequently, the scheme differs in many aspects from standard approaches, for which we refer the reader e.g. to [11, 50]. In particular, due to the staggered discretization, the system is not treated "as a whole", but each equation are updated successively, which makes the numerical analysis different, see e.g. [4, 24, 28, 30, 49]. Next, the definition of the fluxes involves the characteristic speeds of the system, but, despite the "kinetic" motivation, their evaluation do not require to compute complicated integrals. They are defined by simple formula and they do not require additional computational cost. The scheme can be shown to preserve the positivity of the density and the entropy dissipation property under a suitable CFL condition [5], it is thus consistent with the Euler system [4].

Next, we address the full Euler model

$$\partial_t \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \rho E \end{pmatrix} + \boldsymbol{\nabla} \cdot \begin{pmatrix} \rho \mathbf{u} \\ \rho \mathbf{u} \otimes \mathbf{u} + p \mathbb{I} \\ \rho E \mathbf{u} + p \mathbf{u} \end{pmatrix} = 0. \tag{1.2}$$