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Finite-Volume-Particle Methods for the Two-Component Camassa-Holm System

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Abstract. We study the two-component Camassa-Holm (2CH) equations as a model for the long time water wave propagation. Compared with the classical Saint-Venant system, it has the advantage of preserving the waves amplitude and shape for a long time. We present two different numerical methods—finite volume (FV) and hybrid finite-volume-particle (FVP) ones. In the FV setup, we rewrite the 2CH equations in a conservative form and numerically solve it by the central-upwind scheme, while in the FVP method, we apply the central-upwind scheme to the density equation only while solving the momentum and velocity equations by a deterministic particle method. Numerical examples are shown to verify the accuracy of both FV and FVP methods. The obtained results demonstrate that the FVP method outperforms the FV method and achieves a superior resolution thanks to a low-diffusive nature of a particle approximation.

AMS subject classifications: 65M08, 76M12, 76M28, 86-08, 76M25, 35L65

Key words: Two-component Camassa-Holm system, finite-volume method, deterministic particle method, finite-volume-particle method, central-upwind scheme.

1 Introduction

Due to the potential tragic nature of tsunami waves, there is a need for the scientific understanding and modeling of this complicated phenomenon in order to reduce unwanted destruction and prevent unnecessary deaths from this natural disaster. Tsunami

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waves are caused by the displacement of a large volume of a body of water, typically an ocean or a large lake; see, e.g., [6, 48, 49]. They do not resemble other sea waves and are instead characterized by having relatively low amplitude (wave height) offshore, large wavelength, and large characteristic wave speed. This characterization is what prevents tsunami waves from being felt at sea. Tsunami waves grow in height as they reach shallowing water, in what is known as a *wave shoaling* process. In this process, the wave slows down, the wavelength decreases, and a very high and powerful wave arrives on the shore and may cause massive destruction.

There have been many attempts to create accurate models and corresponding numerical methods for simulating tsunami waves. One popular model in the shallow water wave theory is the classical Saint-Venant (SV) system [27], which approximates the behavior of real ocean waves in a reasonable way and is a depth-averaged system that can be derived from the Navier-Stokes equations; see, e.g., [35, 43, 50]. The SV system is widely used to describe flows in lakes, rivers and coastal areas, in which the typical time and space scales of interest are relatively short. Since the SV system is quite difficult to solve, it is sometimes simplified in a number of ways, including linearization, in which the velocity of water particles is taken to be the gradient of a scalar potential. Taking various asymptotic limits of the inviscid Euler equations results in a host of integrable and nearly integrable equations such as the Korteweg-de Vries (KdV) equation, Camassa-Holm (CH) equation, nonlinear Schrödinger equation, and so on; see, e.g., [7, 36, 53, 56]. Unfortunately, while these equations have exact (integrable) solutions, they also diverge from the true behavior described by the full equations for any but very short time scales.

Tsunami waves form in deep water and travel very long distances (thousands of kilometers) before coming to shore. Over long time, solutions of the SV system break down, dissipate in an unphysical manner, develop shock waves, and fails to capture small, trailing waves that are seen in nature and laboratory experiments. Thus, it is necessary to use a more sophisticated model in order to preserve the wave characteristics over long time simulations.

Non-hydrostatic models (the celebrated Green-Naghdi (GN) equations [31] and several others; see, e.g., [1–3] and references therein) work well for long-time propagation of tsunami-type waves because they allow the wave to travel for long distances without changing the shape or decaying in amplitude. In addition, since these systems are dispersive, they give rise to trailing waves that are observed to follow tsunamis in nature. It is, however, necessary to achieve some balance between dispersion observed with a non-hydrostatic model and dissipation seen in the classical SV system.

One attempt to achieve such a balance was made in [4, 5], where the non-hydrostatic SV system was rigorously derived from the GN equations. As it has been demonstrated in [16], the non-hydrostatic SV system is capable of accurately modeling long-time propagation of tsunami-type waves. However, the system is quite complicated and developing accurate, robust and efficient numerical methods for computing its solutions is a highly nontrivial task.

Another system that has been derived from the GN equations is a two-component