Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2018-0202

Vol. **27**, No. 2, pp. 412-430
February 2020

# A Conservative Numerical Method for the Cahn–Hilliard Equation with Generalized Mobilities on Curved Surfaces in Three-Dimensional Space

Darae Jeong[1], Yibao Li[2], Chaeyoung Lee[3], Junxiang Yang[3] and Junseok Kim[3,*]

[1] *Department of Mathematics, Kangwon National University, Gangwon-do 24341, Republic of Korea.*
[2] *School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China.*
[3] *Department of Mathematics, Korea University, Seoul 02841, Republic of Korea.*

Received 8 August 2018; Accepted (in revised version) 4 January 2019

**Abstract.** In this paper, we develop a conservative numerical method for the Cahn–Hilliard equation with generalized mobilities on curved surfaces in three-dimensional space. We use an unconditionally gradient stable nonlinear splitting numerical scheme and solve the resulting system of implicit discrete equations on a discrete narrow band domain by using a Jacobi-type iteration. For the domain boundary cells, we use the trilinear interpolation using the closest point method. The proposing numerical algorithm is computationally efficient because we can use the standard finite difference Laplacian scheme on three-dimensional Cartesian narrow band mesh instead of discrete Laplace–Beltrami operator on triangulated curved surfaces. In particular, we employ a mass conserving correction scheme, which enforces conservation of total mass. We perform numerical experiments on the various curved surfaces such as sphere, torus, bunny, cube, and cylinder to demonstrate the performance and effectiveness of the proposed method. We also present the dynamics of the CH equation with constant and space-dependent mobilities on the curved surfaces.

**AMS subject classifications**: 37M05, 65M06, 65M22, 65P99

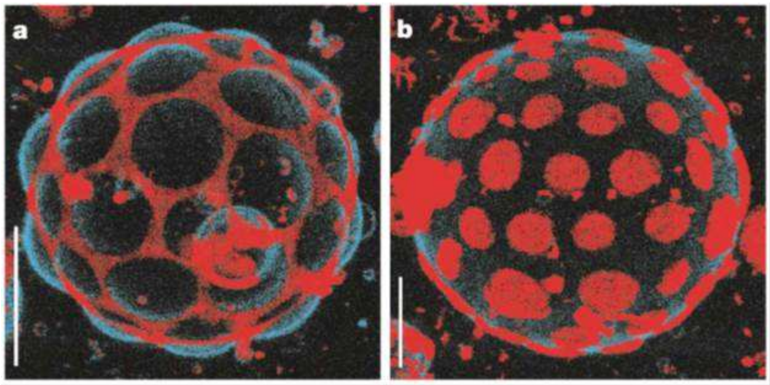
**Key words**: Cahn–Hilliard equation, mass correction scheme, narrow band domain, closest point method.

## 1 Introduction

The Cahn–Hilliard (CH) equation is a fourth-order nonlinear parabolic partial differential equation, originally proposed by Cahn and Hilliard [1, 2] to model phase separation of

*Corresponding author. *Email addresses:* cfdkim@korea.ac.kr (J. Kim), tinayoyo@kangwon.ac.kr (D. Jeong), yibaoli@xjtu.edu.cn (Y. Li), chae1228@korea.ac.kr (C. Lee), nexusxiang@outlook.com (J. Yang)




(a) (b)

Figure 1: Snapshots of arrangement into (a) hexagonal and (b) inverted hexagonal patterns, depending on membrane composition. Reprinted from T. Baumgart *et al.* [21] with permission from Nature Publishing Group.

binary alloys:

$$\frac{\partial\phi(\mathbf{x},t)}{\partial t}=\nabla\cdot\left\{M(\phi(\mathbf{x},t))\nabla[F'(\phi(\mathbf{x},t))-\epsilon^2\Delta\phi(\mathbf{x},t)]\right\}, \tag{1.1}$$

where $\phi$ is the difference of two concentrations, $M$ is a concentration-dependent mobility, $F(\phi)$ is a double-well potential, and $\epsilon$ is a constant related to the interfacial thickness. The CH equation has been applied to model many important phenomena such as tumor growth simulation [3], topology optimization [4, 5], inpainting of binary images [6], volume reconstruction [7], surface diffusion motion [8], phase separation [9, 10], microstructures with elastic inhomogeneity [11, 12], microphase separation of diblock copolymers [13], and multiphase fluid flows [5, 14–18]. For the physical, mathematical, and numerical derivations of the binary CH equation, see [19] and references therein. For the basic principles and practical applications of the CH equation, see [20].

An experimental result has demonstrated that phase separations could occur on curved surfaces such as lipid bilayer membranes [21]. Figs. 1(a) and (b) show arrangements into hexagonal and inverted hexagonal patterns, depending on lipid bilayer membrane composition, respectively. In [22], the authors rigorously analyzed the well-posedness and convergence of a fully discrete finite element method for solving the CH equation on a general surface. In [23], an efficient direct discretization method was developed for solving the Cahn–Hilliard equation on unstructured triangular surfaces. By using a conservation law and transport formulae, the authors derive the CH equation on evolving surfaces [24]. In [25], the authors developed a surface finite element method for the numerical solution of the CH equation on hypersurfaces $\Gamma$ in $\mathbb{R}^3$. The authors in [26] developed a finite difference method for the CH equation on implicit surfaces defined using a level set function.

The main purpose of this article is to develop an efficient conservative finite difference method for the CH equation with generalized mobilities on curved surfaces in three-