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A Shifted-Inverse Adaptive Multigrid Method for the Elastic Eigenvalue Problem

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Abstract. A shifted-inverse iteration is proposed for the finite element discretization of the elastic eigenvalue problem. The method integrates the multigrid scheme and adaptive algorithm to achieve high efficiency and accuracy. Error estimates and optimal convergence for the proposed method are proved. Numerical examples show that the proposed method inherits the advantages of both ingredients and can compute low regularity eigenfunctions effectively.

AMS subject classifications: 65N25, 65N30

Key words: Elastic eigenvalue problem, shifted-inverse iteration, adaptive multigrid method.

1 Introduction

Numerical computation of eigenvalue problems is of fundamental importance in many scientific and engineering applications. Due to the flexibility in treating complex structures and rigorous theoretical justification, finite element methods, including conforming finite elements, mixed finite elements, and discontinuous Galerkin methods, have been popular for eigenvalue problems of partial differential equations [1, 2, 20]. In this paper, we focus on the development of an efficient finite element discretization for the elastic eigenvalue problem on polygonal domains, in particular, when the regularity of the eigenfunctions might be low due to the non-convex domains or discontinuous material properties.

While extensive results have been obtained for the Dirichlet eigenvalue problem, the Maxwell's eigenvalue problem, and the biharmonic eigenvalue problem, much fewer

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works exist in the literature on the elastic eigenvalue problem [9, 10, 13, 16, 18, 19, 22]. In [19], Ovtchinnikov and Xanthis employed a special preconditioning technique associated with the effective dimensional reduction algorithm for the thin elastic structures such as shells, plates and rods of arbitrary geometry. Walsh et al. developed an a posteriori error estimator for heterogeneous elastic structures, which are independent of the variations in material properties and the polynomial degree of finite elements [22]. In [13], Erwin analyzed the finite element approximation of the spectral problem for the linear elasticity equation with mixed boundary conditions on a curved non-convex domain. Using a mixed variational formulation, Meddahi et al. showed that the lowest order Arnold-Falk-Winther element provides a correct approximation of the spectrum with quasi-optimal error estimates [18]. In [9], Russo presented a theory for the approximation of eigenvalue problems in mixed form by non-conforming methods and applied it to the classical Hellinger-Reissner mixed formulation for a linear elastic structure. Recently, Lee et al. used the immersed finite element method based on Crouzeix-Raviart P1-nonconforming element to approximate eigenvalue problems for elasticity equations with interfaces [16]. Domínguez et al. [10] analyzed the Jone's eigenvalue problem, an elastic eigenvalue problem with a special boundary condition, and approximated the eigenvalues using the linear Lagrange element.

The multigrid or multilevel discretization is an efficient tool for eigenvalue problems [11,14,17,23,24]. Recently, it was used with the shifted-inverse iteration and proved to be effective for the Dirichlet eigenvalue problem [25,26], the Maxwell eigenvalue problem, and the integral operator eigenvalue problem [26]. In this paper, we extend the shifted-inverse iteration multigrid method for the elastic eigenvalue problem. To further improve the accuracy and efficiency for lower regularity problems, the adaptive scheme is integrated into the multigrid method. Numerical examples show that the proposed method is highly effective and efficient. For adaptive finite element approximation of eigenvalue problems, we refer the readers to Chp. 8 of [20], [7,8,12,15] and the references therein.

The rest of the paper is organized as follows. In Section 2, we present the elastic eigenvalue problem and finite element convergence results. Section 3 is devoted to the multigrid scheme based on Rayleigh quotient iteration. The convergence results are proved. In Section 4, we present an a posterior error estimate and propose an adaptive method. The algorithm combining the multigrid scheme and the adaptive method is given. Finally, in Section 5, we present several numerical examples on a non-convex domain and of discontinuous material properties, validating the efficiency and accuracy of the proposed method.

2 FEM for the elastic eigenvalue problem

Let $\mathbf{x}=(x,y)^\top\in\mathbb{R}^2$, $\Omega\subset\mathbb{R}^2$ be a bounded Lipschitz polygon, and $\mathbf{H}_0^1(\Omega):=H_0^1(\Omega)\times H_0^1(\Omega)$, $\mathbf{L}^2(\Omega):=L^2(\Omega)\times L^2(\Omega)$. The elastic eigenvalue problem is to find $(\lambda,\mathbf{u})\in\mathbb{R}\times\mathbf{H}_0^1(\Omega)$ such